Review for final – solutions

① Arc length of $x(t)=e^{-t}\cos t$, $y(t)=e^{-t}\sin t$, $0\le t\le \frac{\pi}{2}$.

$$s=\int_a^b\sqrt{(x'(t))^2+(y'(t))^2}\,dt=\int_0^{\pi/2}\sqrt{\left(-e^{-t}(\underline{\sin t+\cos t})\right)^2+\left(e^{-t}(\underline{\cos t-\sin t})\right)^2}\,dt$$

$x'(t)=-e^{-t}\sin t-e^{-t}\cos t=-e^{-t}(\sin t+\cos t)$

$y'(t)=e^{-t}\cos t-e^{-t}\sin t=e^{-t}(\cos t-\sin t)$

$$=\int_0^{\pi/2}\sqrt{e^{-2t}(\sin^2 t+\cancel{2\sin t\cos t}+\cos^2 t+\cos^2 t-\cancel{2\sin t\cos t}+\sin^2 t)}\,dt$$

$$=\int_0^{\pi/2}e^{-t}\sqrt{2}\,dt=\left[-\sqrt{2}e^{-t}\right]_0^{\pi/2}$$

$$=-\sqrt{2}\left[e^{-\pi/2}-1\right]=\boxed{\sqrt{2}\,(1-e^{-\pi/2})}$$

② $\bar{u}=\langle 1,-2,3\rangle$, $\bar{v}=\langle 2,5,-1\rangle$

A) $\bar{u}\cdot\bar{v}=2-10-3=-11$

B) $\bar{u}\cdot\bar{v}=\|\bar{u}\|\,\|\bar{v}\|\cos\theta$, $-11=\sqrt{1+4+9}\sqrt{4+25+1}\,\cos\theta$,

$-11=\sqrt{14}\sqrt{30}\,\cos\theta$, $\cos\theta=\frac{-11}{\sqrt{420}}=-.5367\Rightarrow\theta=\boxed{122^\circ}$

C) $\text{proj}_{\bar{u}}\bar{v}=\frac{\bar{u}\cdot\bar{v}}{\|\bar{u}\|^2}\bar{u}=\frac{-11}{1+4+9}\langle 1,-2,3\rangle=\boxed{\frac{-11}{14}\langle 1,-2,3\rangle}$

③ Area of Δ with verticies $P(1,3,5)$, $Q(3,3,0)$, $R(-2,0,5)$

$\overrightarrow{PQ}=\langle 2,0,-5\rangle$ $\quad\overrightarrow{PR}=\langle -3,-3,0\rangle$

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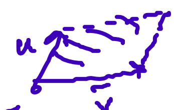


$\|\bar{u}\times\bar{v}\|$

Area $=\frac{1}{2}\|\overrightarrow{PQ}\times\overrightarrow{PR}\|=$

$$\|\overrightarrow{PQ}\times\overrightarrow{PR}\|=\left\|\begin{vmatrix}0&-5\\-3&0\end{vmatrix}\bar{i}-\begin{vmatrix}2&-5\\-3&0\end{vmatrix}\bar{j}+\begin{vmatrix}2&0\\-3&-3\end{vmatrix}\bar{k}\right\|$$

$$=\left\|(-15)\bar{i}-(-15)\bar{j}+(-6)\bar{k}\right\|=\sqrt{225+225+36}=\sqrt{486}=9\sqrt{6}$$

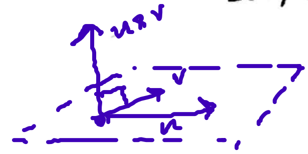
$\Rightarrow$ area of Δ $=\boxed{\frac{1}{2}9\sqrt{6}}$.

④ Equa of line thru $(1,2,3)$ and $\parallel$ to $\underline{\langle 2,0,-1\rangle}$.

$x=2t+1$

$y=0t+2$

$z=-t+3$



(5) Equa of plane thru $(0,0,0)$, $(1,2,3)$, $(-2,3,3)$.

let $\bar{u} = \langle 1-0, 2-0, 3-0 \rangle = \langle 1,2,3 \rangle$

$\bar{v} = \langle -2-0, 3-0, 3-0 \rangle = \langle -2,3,3 \rangle$

$$\bar{u} \times \bar{v} = \begin{vmatrix} 2 & 3 \\ 3 & 3 \end{vmatrix} \bar{i} - \begin{vmatrix} 1 & 3 \\ -2 & 3 \end{vmatrix} \bar{j} + \begin{vmatrix} 1 & 2 \\ -2 & 3 \end{vmatrix} \bar{k}$$

$$= (6-9)\bar{i} - (3+6)\bar{j} + (3+4)\bar{k}$$

$$= \underline{-3}\bar{i} - \underline{9}\bar{j} + \underline{7}\bar{k} \quad \leftarrow \text{vector} \perp \text{to plane}$$

Equa: $-3(x-0) - 9(y-0) + 7(z-0) = 0$

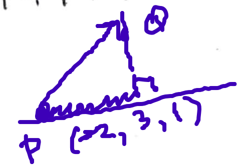
$-3x - 9y + 7z = 0$ or

$$\boxed{3x + 9y - 7z = 0}$$

(6) Dist from $P(1,5,-4)$ to $\overset{a}{3}x \overset{b=-1}{-} y + \overset{c}{2}z = 6$ $\quad d=-6$

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2+b^2+c^2}} = \frac{|3(1) - (5) + 2(-4) - 6|}{\sqrt{9+1+4}}$$

$$= \boxed{\frac{16}{\sqrt{14}}}$$



(7) Dist from $Q(10,3,-2)$ to $x = 4t-2$; $y=3$; $z = -t+1$.

Let P be a point on the line. Choose $P = (-2,3,1)$.

$\vec{PQ} = \langle 12, 0, -3 \rangle$. let $\bar{u} = \langle 4, 0, -1 \rangle$ (direction vector)

$$\text{Dist} = \|\vec{PQ}\| \sin\theta = \frac{\|\vec{PQ} \times \bar{u}\|}{\|\bar{u}\|}$$

$\vec{PQ} = \langle 12, 0, -3 \rangle$
$\vec{u} = \langle 4, 0, -1 \rangle$

$$\vec{PQ} \times \vec{u} = \begin{vmatrix} 0 & -3 \\ 0 & 1 \end{vmatrix} \bar{i} - \begin{vmatrix} 12 & -3 \\ 4 & -1 \end{vmatrix} \bar{j} + \begin{vmatrix} 12 & 0 \\ 4 & 0 \end{vmatrix} \bar{k} =$$

$$= 0\bar{i} - (-12+12)\bar{j} + (0)\bar{k} = \vec{0}$$

So, $\text{Dist} = \dfrac{0}{\|\bar{u}\|} = \boxed{0}$

(8) $\frac{x^2}{4} + \frac{z^2}{16} - \frac{y^2}{9} = -1$

$\underbrace{\frac{x^2}{4} + \frac{z^2}{16}}_{\geq 0} = \frac{y^2}{9} - 1 \leftarrow \geq 0 \Rightarrow y^2 - 9 \geq 0$

$+ \quad - \quad +$; -3, 3 $\Rightarrow$ $y \leq -3$ or $y \geq 3$

In yz-plane : $x=0$ $\Rightarrow$ $\frac{z^2}{16} - \frac{y^2}{9} = -1$ or $\frac{y^2}{9} - \frac{z^2}{16} = 1$

axis is y-axis.

(9) $f(x,y) = \frac{1}{\sqrt{(x-1)(y+3)}}$; $(x-1)(y+3) > 0$

$\Rightarrow$ $x-1>0$ and $y+3>0$ $\boxed{x>1 \text{ and } y>-3}$ or $x-1<0$, $y+3<0$ $\boxed{x<1 \text{ and } y<-3}$

(10) $f(x,y) = xy^3 - x^2 + y^2$

A) $f_x = y^3 - 2x$

$f_{xy} = 3y^2$

$f_y = 3xy^2 + 2y$

$3(1)(2)^2 + 4$

$f_y(1,2) = 3(1)(4) + (2)(2)$
$= 12 + 4 = 16$

* B) tangent line to $f(x,y)$ at $(1,2)$ in the plane $y=2$.

$f_x(1,2) = (2)^3 - 2(1) = 8-2 = 6 \leftarrow$ slope of line

$\Rightarrow$ tangent vector to $f(x,y)$ in plane $y=2$ is $\langle 1, 0, 6\rangle$

x inc by 1 ; z inc by 6

Line thru $(1,2,f(1,2)) = ((1)(8)-(1)+(4))$
$= (1,2,11)$

so tangent line is : $x(t) = 1+t$
$y(t) = 2 + 0t$
$z(t) = 11 + 6t$

gradient

C) $D_{\bar{u}} f(1,2)$ where $\bar{u} = \langle 1,0\rangle$ is

$\langle f_x(1,2), f_y(1,2)\rangle \cdot \langle 1,0\rangle =$

$= \langle 6, 16\rangle \cdot \langle 1,0\rangle = \boxed{6}$

diff: $f_x\,dx + f_y\,dy$

D) Diff at $(1,2)$ where $dx = .1$ and $dy = .2$

$f_x(1,2)(.1) + f_y(1,2)(.2) = (6)(.1) + (16)(.2) = .6 + 3.2 = \boxed{3.8}$

(10) E) the slope of the curve at point (1,2) in the direction of $\bar{u} = \langle \frac{3}{5}, \frac{4}{5} \rangle$: ← $\|\bar{u}\|=1$

$$D_u(1,2) = \langle f_x(1,2), f_y(1,2) \rangle \cdot \langle \tfrac{3}{5}, \tfrac{4}{5} \rangle$$

$$= \langle 6, 16 \rangle \cdot \langle \tfrac{3}{5}, \tfrac{4}{5} \rangle = \frac{18}{5} + \frac{64}{5} = \boxed{\frac{82}{5}} \qquad \frac{82}{5}$$

(11) $f(x,y) = 2x^2 + 2xy + y^2 + 2x - 3$ { max/min saddle

$f_x = 4x + 2y + 2$ $\qquad$ $f_y = 2x + 2y$

$f_{xx} = 4$ $\qquad$ $f_{yy} = 2$

$f_{xy} = 2$

$f_x = 4x + 2y + 2 = 0$ $\qquad$ $f_y = 2x + 2y = 0$

$\Rightarrow y = -x$

$4x + 2(-x) + 2 = 0$

$2x + 2 = 0$

$2x = -2$

$x = -1$ ← $(-1, 1)$ cv

$\Rightarrow y = 1$

At $x = -1$, $y = 1$

$D = f_{xx} f_{yy} - [f_{xy}]^2$

$D\big|_{(-1,1)} = f_{xx}(-1,1)\, f_{yy}(-1,1) - [f_{xy}(-1,1)]^2$ $\qquad \cup$

$= (4)(2) - (2)^2 = 8 - 4 = 4 > 0$

$\Rightarrow$ Min at $(-1,1)$

sol/5

(12) $f(x,y) = x^2 + y^2 + xy$

A) $x(t) = 5t+1 \qquad y(t) = 7t^2$

$$\frac{df}{dt} = f_x\, x'(t) + f_y\, y'(t) = (2x+y)\underbrace{(5)}_{x'(t)} + (2y+x)\underbrace{(14t)}_{y'(t)}$$

B) $x(s,t) = s^2+3t^2 \;;\; y(s,t) = 7s+5t$

$$\frac{\partial f}{\partial s} = f_x \frac{\partial x}{\partial s} + f_y \frac{\partial y}{\partial s} = (2x+y)(2s) + (2y+x)(7)$$

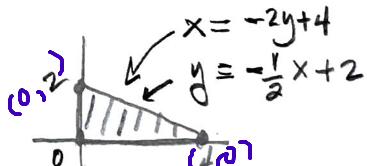
(13) Cone: $V = \frac{1}{3}\pi r^2 h \;;\; h(t) = 5t,\; r(t) = 7t^2$. Find $\frac{dV}{dt}$ when $t=1$.

$$\frac{dV}{dt} = V_r \frac{dr}{dt} + V_h \frac{dh}{dt} = \left(\frac{2}{3}\pi r h\right)(14t) + \left(\frac{1}{3}\pi r^2\right)(5)$$

when $t=1$, $\frac{dV}{dt} = \left(\frac{2}{3}\pi(7)(5)\right)(14) + \left(\frac{1}{3}\pi(7)^2\right)(5)$

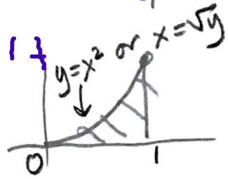
$$= \frac{70\pi}{3} + \frac{245\pi}{3} = \boxed{\frac{315\pi}{3}}$$

(14)



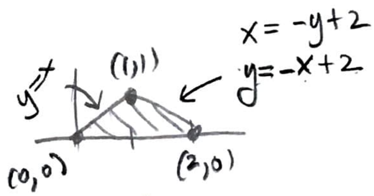
$$\int_0^4 \int_0^{-\frac{1}{2}x+2} f(x,y)\,dy\,dx \quad \text{or} \quad \int_0^2 \int_0^{-2y+4} f(x,y)\,dx\,dy$$

(15)



$$\int_0^1 \int_0^{x^2} 1\,dy\,dx \quad \text{or} \quad \int_0^1 \int_{\sqrt{y}}^1 1\,dx\,dy$$

(16)



$$\int_0^1 \int_y^{-y+2} x^2+y^2\,dx\,dy = \frac{4}{3}$$

(17) A) $\sum_1^\infty \frac{n^2}{\sqrt{n^8+1}}$ $\quad \frac{n^2}{\sqrt{n^8+1}} \le \frac{n^2}{\sqrt{n^8}} = \frac{n^2}{n^4} = \frac{1}{n^2}$ $\quad$ direct comp $(\le)$ with $\sum \frac{1}{n^2}$ conv ✓ $\Rightarrow$ conv.

B) $\sum_1^\infty \frac{n}{n^3-1}$ $\quad \frac{n}{n^3-1} \sim \frac{n}{n^3} = \frac{1}{n^2}$ $\quad$ $\sum \frac{1}{n^2}$ conv $\Rightarrow$ conv

Limit comp: $\frac{\frac{1}{n^2}}{\frac{n}{n^3-1}} = \frac{n^3-1}{n} \cdot \frac{1}{n^2} = \frac{n^3-1}{n^3} = \frac{1-\frac{1}{n^3}}{1} \to 1 > 0 \Rightarrow$ conv by Lim Comp

C) $\sum_1^\infty \frac{(-1)^n}{2n+1}$ $\quad$ Alt series; $a_n \to 0$: $\frac{1}{2n+1} \to 0$ $\quad a_{n+1} \le a_n$: $\frac{1}{2(n+1)+1} \le \frac{1}{2n+1}$ $\Rightarrow$ conv.

D) $\sum_1^\infty n^2 \left(\frac{2}{3}\right)^n$ $\quad$ Ratio test: $\left| \frac{(n+1)^2 \left(\frac{2}{3}\right)^{n+1}}{n^2 \left(\frac{2}{3}\right)^n} \right| = \underbrace{\left(\frac{n+1}{n}\right)^2}_{\to 1} \left(\frac{2}{3}\right) \to \frac{2}{3}$ $\quad 0 < \frac{2}{3} < 1$ $\Rightarrow$ conv by Ratio test ✓

(18) $\sum_{4}^{\infty}\left(\frac{2}{3}\right)^n = \left(\frac{2}{3}\right)^4+\left(\frac{2}{3}\right)^5+\cdots = \left(\frac{2}{3}\right)^4\left[1+\left(\frac{2}{3}\right)+\left(\frac{2}{3}\right)^2+\cdots\right]$ sol/6

$$= \left(\frac{2}{3}\right)^4 \frac{1}{1-\frac{2}{3}} = \left(\frac{2}{3}\right)^4(3) = .59259$$

(19) Int. of conv: (use Ratio test)

A) $\sum_{0}^{\infty}\frac{(x-2)^n}{10^n}$ $\quad \left|\frac{(x-2)^{n+1}}{10^{n+1}}\cdot\frac{10^n}{(x-2)^n}\right| = \left|\frac{x-2}{10}\right| < 1 \Rightarrow |x-2| < 10$

$-10 < x-2 < 10$

$-8 < x < 12$

check endpts:

$x=12$ $\quad \sum_{0}^{\infty}\frac{(10)^n}{(10)^n} = \sum_{0}^{\infty}(1)^n$ diverges

$\boxed{(-8,12)}$

$x=-8$ $\quad \sum_{0}^{\infty}\frac{(-10)^n}{(10)^n} = \sum_{0}^{\infty}(-1)^n$ diverges

B) $\sum_{0}^{\infty}\frac{nx^n}{n+2}$ $\quad \left|\frac{(n+1)x^{n+1}}{n+3}\right|\cdot\left|\frac{n+2}{nx^n}\right| = \left|\overset{\to 1}{\frac{(n+1)}{n}}\cdot\overset{\to 1}{\frac{n+2}{n+3}}\,x\right| \to |x|$

$|x|<1$

endpts:

$\sum_{0}^{\infty}\frac{n}{n+2}(1)^n$ div $\quad (a_n \not\to 0)$ DIV

$\boxed{(-1,1)}$

$\sum_{0}^{\infty}\frac{n}{n+2}(-1)^n$ div $\quad (a_n \not\to 0)$ DIV

C) $\sum_{1}^{\infty}\frac{x^n}{n\sqrt{n}\,3^n}$ $\quad \left|\frac{x^{n+1}}{(n+1)\sqrt{n+1}\,3^{n+1}}\right|\cdot\left|\frac{n\sqrt{n}\,3^n}{x^n}\right| = \left|\overset{\to 1}{\frac{n}{n+1}}\,\overset{\to 1}{\frac{\sqrt{n}}{\sqrt{n+1}}}\,x\,\frac{1}{3}\right|$

$\to \left|\frac{x}{3}\right| < 1$

$\Longrightarrow |x| < 3$

$-3 < x < 3$

end pts

$x=3$ $\quad \sum_{1}^{\infty}\frac{3^n}{n\sqrt{n}\,3^n} = \sum_{1}^{\infty}\frac{1}{n^{3/2}}$ conv by p-test

$\boxed{[-3,3]}$

$x=-3$ $\quad \sum_{1}^{\infty}\frac{(-1)^n}{n\sqrt{n}}$ conv, Alt series $a_n\to 0$, $a_{n+1}\le a_n$

D) $\sum_{0}^{\infty}\frac{x^n}{\sqrt{n^2+3}}$ $\quad \left|\frac{x^{n+1}}{\sqrt{(n+1)^2+3}}\cdot\frac{\sqrt{n^2+3}}{x^n}\right| = \left|\overset{\to 1}{\frac{\sqrt{n^2+3}}{\sqrt{(n+1)^2+3}}}\cdot x\right| \to |x| < 1$ $\quad -1 < x < 1$

endpts: $x=1$ $\sum\frac{1}{\sqrt{n^2+3}} \sim \sum\frac{1}{n}$ div ; $x=-1$ $\sum\frac{(-1)^n}{\sqrt{n^2+3}}$ conv $\Rightarrow \boxed{[-1,1)}$

lim comp

(20) use $\sum_{0}^{\infty}(-.3)^n$ to obtain est of sum with an accuracy of (.01).

We have an <u>alt</u> <u>series</u>

$(.3)^4 = .008 < .01 \quad \Rightarrow \quad$ use $1 + (-.3) + (-.3)^2 + (-.3)^3 = \boxed{.763}$

(21) $$\frac{1}{5x+3} = \frac{1/3}{1+\frac{5}{3}x} = \frac{1/3}{1-\left(-\frac{5x}{3}\right)} = \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{5x}{3}\right)^n$$

converges for $\left|\frac{5x}{3}\right| < 1 \quad \Rightarrow \quad |x| < \frac{3}{5}$

$-\frac{3}{5} < x < \frac{3}{5}$

Diverges for $x = \frac{3}{5}, -\frac{3}{5}$

$\boxed{\left(-\frac{3}{5}, \frac{3}{5}\right)}$

(22) $f(x) = \sum_{1}^{\infty} \frac{x^n}{n^2}$, find $F(x)$, $F'(x) = f(x)$, $F(0) = 2$

$$F(x) = \sum_{1}^{\infty} \frac{x^{n+1}}{n^2(n+1)} + C \qquad F(0) = 2 \Rightarrow C = 2$$

C = 2

(23) $$f(x) = e^x \sin x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$= x$

$+ x^2$

$+ \frac{-x^3}{3!} + \frac{x^3}{2} \quad \leftarrow \quad \frac{-x^3}{6} + \frac{3x^3}{6}$

$+ \frac{-x^4}{3!}$

$= \boxed{x + x^2 + \frac{1}{3}x^3 + \cdots}$