Math 21 Review Exam 3 – SOLUTIONS

(1) $$.151515\ldots = \frac{15}{100} + \frac{15}{100^2} + \frac{15}{100^3} + \cdots =$$

$$\frac{15}{100}\left(1 + \frac{1}{100} + \frac{1}{100^2} + \cdots\right) = \frac{15}{100} \cdot \frac{1}{1-\frac{1}{100}} = \frac{15}{100}\,\frac{1}{\frac{99}{100}} = \frac{15}{99} = \boxed{\frac{5}{33}}$$

(2) $$\sum_{10}^{\infty} 5(.34)^n = 5\left[(.34)^{10} + (.34)^{11} + (.34)^{12} + \cdots\right]$$

$$= 5(.34)^{10}\left[1 + (.34) + (.34)^2 + \cdots\right] = 5(.34)^{10}\,\frac{1}{1-.34} = \boxed{.000156}$$

(3) $$\sum_{1}^{\infty} \frac{-2}{n^2+2n} = \sum_{1}^{\infty} \frac{-2}{n(n+2)}$$

$$\frac{-2}{n(n+2)} = \frac{A}{n+2} - \frac{B}{n}$$

$$-2 = A(n+2) + Bn$$

$$-2 = -2B \Rightarrow B = 1$$

$$-2 = 2A \Rightarrow A = -1$$

$$\sum_{1}^{\infty} \frac{-2}{n^2+2n} = \sum_{1}^{\infty}\left(\frac{1}{n+2} - \frac{1}{n}\right) =$$

$$\left(\frac{1}{3} - \frac{1}{1}\right) + \left(\frac{1}{4} - \frac{1}{2}\right) + \left(\frac{1}{5} - \frac{1}{3}\right) + \left(\frac{1}{6} - \frac{1}{4}\right) + \left(\frac{1}{7} - \frac{1}{5}\right) + \cdots$$

$$= -1 - \frac{1}{2} = \boxed{-\frac{3}{2}}$$

(4) A) $$\sum_{1}^{\infty} \frac{n^2}{\sqrt{n^3+1}}\ ; \quad \frac{n^2}{\sqrt{n^3+1}} = \frac{1}{\sqrt{\frac{1}{n} + \frac{1}{n^4}}} \xrightarrow[\lim_{n\to\infty}]{} \frac{1}{0} \Rightarrow \text{Diverges}$$

B) $$\sum_{1}^{\infty} \frac{(-1)^n}{\sqrt{n}}\ ;$$ Alternating Series

$$\frac{1}{\sqrt{n+1}} \le \frac{1}{\sqrt{n}} \text{ and } \frac{1}{\sqrt{n}} \xrightarrow[\lim_{n\to\infty}]{} 0 \Rightarrow \text{converges}$$

$$f(x) = x^{-1/2}$$

$$f'(x) = -\frac{1}{2}x^{-3/2} < 0 \Rightarrow \text{decreases}$$

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c) $\sum_{0}^{\infty} \frac{n}{2^n}$; $\left|\frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n}\right| = \left|\frac{n+1}{n} \cdot \frac{1}{2}\right| \xrightarrow[n\to\infty]{\lim} \frac{1}{2} < 1$

$\Rightarrow$ converges by ratio test.

D) $\sum_{1}^{\infty} \frac{1}{n^{3/2}}$ converges by the p-test $(p = \frac{3}{2} > 1)$

or $\int_1^{\infty} \frac{1}{x^{3/2}}\,dx$ converges.

E) $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$ converges by the integral test.

$$\int_2^{\infty} \frac{1}{x}(\ln(x))^{-2}\,dx = \left[-(\ln(x))^{-1}\right]_2^{\infty} = \lim_{R\to\infty} \frac{-1}{\ln(R)} + \frac{1}{\ln(2)}$$

$$= \boxed{\frac{1}{\ln(2)}} \Rightarrow \Sigma \text{ converges.}$$

F) $\sum_{1}^{\infty} \frac{n}{\sqrt{n^4+2}}$ ← limit comparison test with $\frac{n}{\sqrt{n^4}} = \frac{n}{n^2} = \frac{1}{n}$

$$\frac{\frac{1}{n}}{\frac{n}{\sqrt{n^4+2}}} = \frac{1}{n} \cdot \frac{\sqrt{n^4+2}}{n} = \frac{\sqrt{n^4+2}}{n^2} = \sqrt{1+\frac{2}{n^4}} \xrightarrow[n\to\infty]{\lim} \sqrt{1},$$

$0 < 1 < \infty \Rightarrow$ diverges by limit comparison test since $\Sigma \frac{1}{n}$ diverges (by p-test).

G) $\sum_{1}^{\infty} \frac{n}{\sqrt{n^5+1}}$

Direct comparison test with $\sum_{1}^{\infty} \frac{n}{\sqrt{n^5}} = \sum_{1}^{\infty} \frac{1}{n^{3/2}}$.

Now $\Sigma \frac{1}{n^{3/2}}$ conv by the p-test and $\frac{n}{\sqrt{n^5+1}} \leq \frac{n}{\sqrt{n^5}}$

$\Rightarrow \Sigma \frac{n}{\sqrt{n^5+1}}$ also converges.

(4)

H) $\sum_{2}^{\infty} \frac{(-1)^n}{\ln(n)}$ is an alternating series;

$\frac{1}{\ln(n)} \xrightarrow[\lim_{n\to\infty}]{} 0$; $\frac{1}{\ln(n+1)} \le \frac{1}{\ln(n)}$ we see this

since $f(x) = \frac{1}{\ln x}$ is decreasing, $f'(x) = \frac{-1}{(\ln x)^2} \cdot \frac{1}{x} < 0$

for $x \geq 2$.

I) $\sum_{1}^{\infty} \frac{n^n}{n!}$

Ratio test;

$\left(1+\frac{1}{n}\right)^n \to e > 1$

$$\frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \left(\frac{n+1}{n}\right)^n \frac{n+1}{n+1} \xrightarrow[\lim_{n\to\infty}]{} e \Rightarrow \text{ratio test concluded series diverges}$$

Also However: $\lim_{n\to\infty} \frac{n^n}{n!} \neq 0 \Rightarrow$ diverges.

$$\lim_{n\to\infty} \frac{n^n}{n!} = \lim_{n\to\infty} \underbrace{\frac{n}{n} \cdot \frac{n}{n-1} \cdot \frac{n}{n-2} \cdots \frac{n}{2} \cdot \frac{n}{1}}_{\text{each factor is } \geq 1} \neq 0.$$

J) $\sum_{1}^{\infty} \frac{2^{4n}}{(2n+1)!}$

Ratio test:

$$\frac{2^{4n+4}}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{2^{4n}} = \frac{2^4}{1} \cdot \frac{1}{(2n+3)(2n+2)} \xrightarrow[\lim_{n\to\infty}]{} 0 < 1$$

$\Rightarrow$ converges.

K) $\sum_{1}^{\infty} \frac{1}{n^2}\left(\frac{3}{2}\right)^n$

Ratio test:

$$\frac{\left(\frac{3}{2}\right)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{\left(\frac{3}{2}\right)^n} = \left(\frac{n}{n+1}\right)^2 \left(\frac{3}{2}\right) \xrightarrow[\lim_{n\to\infty}]{} \frac{3}{2} > 1 \Rightarrow \text{Diverges}$$

⑤ Deg 2 Taylor poly of $f(x) = x^2\cos x$ about $c = \pi$.

$f(x) = x^2\cos x$ $\qquad$ $f(\pi) = (\pi)^2\cos(\pi) = -\pi^2$

$f^{(1)}(x) = -x^2\sin x + 2x\cos x$ $\qquad$ $f^{(1)}(\pi) = -2\pi$

$f^{(2)}(x) = -x^2\cos x - 2x\sin x$
$\qquad + 2x\sin x + 2\cos x$ $\qquad$ $f^{(2)}(\pi) = \pi^2 - 2$

$$P_2(x) = -\pi^2 - 2\pi(x-\pi) + (\pi^2-2)\frac{(x-\pi)^2}{2}$$

⑥ Interval of convergence ?

A) $\sum_1^\infty (-1)^n \frac{x^n}{5^n}$

$$\left|\frac{x^{n+1}}{5^{n+1}} \cdot \frac{5^n}{x^n}\right| = \left|\frac{x}{5}\right| < 1 \Rightarrow |x| < 5$$
$R = 5$

when $x = 5$ $\quad \sum_1^\infty (-1)^n$ div

when $x = -5$ $\quad \sum_1^\infty (-1)^{2n}$ div

$\Rightarrow$ interval of convergence is $(-5, 5)$.

⑦ Power series for $f(x) = \frac{1}{1-x^2}$ [illegible]

$$\frac{1}{1-x^2} = 1 + x^2 + [illegible] + \ldots^3 + \ldots = \sum_{n=0}^\infty (x)^{2n}$$

$$\left|\frac{x^{2n+2}}{x^{2n}}\right| = |x^2| < 1 \qquad x^2 - 1 < 0$$

pos, neg, pos (-1, 1) $\Rightarrow$ int of conv is $(-1, 1)$.

At $x = 1$ and $x = -1$ $\quad \frac{1}{1-x^2}$ DNE.

(6B) $\sum_{1}^{\infty} \frac{(-1)^n (x-1)^n}{n 5^n}$

$$\left|\frac{(x-1)^{n+1}}{(n+1)5^{n+1}} \cdot \frac{n 5^n}{(x-1)^n}\right| = \left|(x-1)\ \frac{n}{n+1} \cdot \frac{1}{5}\right| \xrightarrow[n\to\infty]{\lim} \left|\frac{x-1}{5}\right| < 1$$

$|x-1| < 5$
$-5 < x-1 < 5$
$-4 < x < 6$

For $x = -4$, $\sum \frac{(-1)^n (-5)^n}{n (5)^n} = \sum \frac{(-1)^{2n}}{n} = \sum \frac{1}{n}$

diverges by p-test

For $x = 6$, $\sum_{1}^{\infty} (-1)^n \frac{1}{n}$ converges, alt series, $\frac{1}{n} \to 0$, $\frac{1}{n+1} \leq \frac{1}{n}$.

$\Rightarrow$ interval of convergence $\boxed{(-4, 6]}$.

c) $\sum_{0}^{\infty} \frac{(x-1)^{n+1}}{(n+1)3^{n+1}}$

$$\left|\frac{(x-1)^{n+2}}{(n+2)3^{n+2}} \cdot \frac{(n+1)3^{n+1}}{(x-1)^{n+1}}\right| = \left|(x-1) \cdot \frac{n+1}{n+2} \cdot \frac{1}{3}\right| \xrightarrow[n\to\infty]{\lim} \left|\frac{x-1}{3}\right| < 1$$

$|x-1| < 3$, $-3 < x-1 < 3$
$-2 < x < 4$

when $x = -2$ $\sum \frac{(-1)^{n+1}}{n+1}$ } converges

when $x = 4$ $\sum \frac{1}{n+1}$ } diverges

$\Rightarrow$ interval of conv: $[-2, 4)$

D) $\sum \frac{(-1)^n x^{2n}}{n!}$; $\left|\frac{x^{2n+2}}{(n+1)!} \cdot \frac{n!}{x^{2n}}\right| = \left|\frac{x^2}{n+1}\right| \xrightarrow[n\to\infty]{\lim} 0 \Rightarrow$ conv $(-\infty, \infty)$

E) $\sum_{1}^{\infty} \frac{n!\, x^n}{(2n)!}$; $\left|\frac{(n+1)!\, x^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{n!\, x^n}\right| = \left|\frac{(n+1)x}{(2n+2)(2n+1)}\right| \to 0 \Rightarrow$ conv $(-\infty, \infty)$

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(7) $f(x) = \frac{1}{1-x^2}$ } power series

$$\frac{1}{1-x^2} = 1 + (x^2) + (x^2)^2 + (x^2)^3 + \cdots = \sum_{n=0}^{\infty} x^{2n}$$

$\left|\frac{x^{2n+2}}{x^{2n}}\right| = |x^2| < 1$ $\quad x^2 - 1 < 0$ $\quad$ + -1 − 1 + $\Rightarrow$ int. of conv. is $(-1, 1)$

(div at $x = \pm 1$, $f(x)$ DNE at $x = \pm 1$)

(8) $f(x) = \frac{1}{5x+3}$ ← power series centered at $c=0$

$$\frac{1}{5x+3} = \frac{1/3}{\frac{5}{3}x+1} = \frac{1/3}{1-\left(-\frac{5x}{3}\right)} = \frac{1}{3}\left[1-\left(\frac{5x}{3}\right)+\left(\frac{5x}{3}\right)^2-\left(\frac{5x}{3}\right)^3+\cdots\right]$$

Geometric series, converges to $\left|\frac{5x}{3}\right|<1$

$|x| < \frac{3}{5}$

$x = \frac{3}{5}$

$\frac{1}{3}[1-1+1-1+1\cdots]$ div → $\boxed{\left(-\frac{3}{5}, \frac{3}{5}\right)}$ ← interval (when $x = -\frac{3}{5}$ denom$=0$)

(9) $$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + \cdots = \sum_0^{\infty} (-1)^n x^{2n}$$

$$\arctan(x) = \int \frac{1}{1+x^2}\,dx = \sum_0^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C$$

$\arctan(0) = 0 \Rightarrow C = 0.$

(10) $f(x) = (1+x)^{1/2}$ ← first four terms of Taylor series

$f(x) = (1+x)^{1/2}$ $\quad f(0) = 1$

$f^{(1)}(x) = \frac{1}{2}(x+1)^{-1/2}$ $\quad f^{(1)}(0) = \frac{1}{2}$

$f^{(2)}(x) = -\frac{1}{2^2}(x+1)^{-3/2}$ $\quad f^{(2)}(0) = -\frac{1}{2^2}$

$f^{(3)}(x) = \frac{+3}{2^3}(x+1)^{-5/2}$ $\quad f^{(3)}(0) = \frac{+3}{2^3}$

$$f(x) \sim \boxed{1 + \frac{1}{2}x - \frac{1}{2^2}\frac{x^2}{2} + \frac{3}{2^3}\frac{x^3}{3!}}$$

(11) Four terms Power series of $f(x) = \sin x \cos x$ centered at $c = 0$.

$f(x) = \sin x \cos x$ $\qquad f(0) = 0$

$f^{(1)}(x) = -\sin x \cdot \sin x + \cos x \cdot \cos x$ $\qquad f^{(1)}(0) = 1$

$= -\sin^2 x + \cos^2 x$

$f^{(2)}(x) = -2\sin x \cdot \cos x + 2\cos x \sin x$ $\qquad f^{(2)}(0) = 0$

$= -4\sin x \cos x$

$f^{(3)}(x) = -4(-\sin^2 x + \cos^2 x)$ $\qquad f^{(3)}(0) = -4$

$f^{(4)}(x) = -4(-4\sin x \cos x)$ $\qquad f^{(4)}(0) = 0$

$= 16 \sin x \cos x$

$f^{(5)}(x) = 16(-\sin^2 x + \cos^2 x)$ $\qquad f^{(5)}(0) = 16$

$f^{(6)}(x) = 16(-4\sin x \cos x)$ $\qquad f^{(6)}(0) = 0$

$= -64 \sin x \cos x$

$f^{(7)}(x) = -64(-\sin^2 x + \cos^2 x)$ $\qquad f^{(7)}(0) = -64$

$$f(x) \sim x - \frac{4}{3!}x^3 + \frac{16}{5!}x^5 - \frac{64}{7!}x^7$$

(12) Power series for e^{-x^2}

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{8!} - \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$

(13) Find the first 3 non-zero terms of power series for $f(x) = e^x \cdot \ln(1+x)$.

$$e^x \ln(1+x) = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right)$$

$$= (x) + \left(x^2 - \frac{x^2}{2}\right) + \left(\frac{x^3}{3} - \frac{x^3}{2} + \frac{x^3}{2}\right)$$

(14)

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots$$

$$e^{-1} = 1 - 1 + \frac{1}{2} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + - \cdots$$

$\frac{1}{n!} < .001$?

$\frac{1}{5!} = \frac{1}{120} = .008$

$\frac{1}{6!} = .001388$

$\frac{1}{7!} = .00019 < .001$

$$e^{-1} \cong 1 - 1 + \frac{1}{2} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}$$

$$= \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720}$$

$$= \frac{360}{720} - \frac{120}{720} + \frac{30}{720} - \frac{6}{720} + \frac{1}{720}$$

$$= \frac{265}{720} = \boxed{.36805}$$

$(e^{-1} = .367879)$

(15) Given that $f(x) = \sum_{0}^{\infty} \frac{(3x)^n}{n+1}$, find $F(x)$, where

$F'(x) = f(x)$ and $F(0) = 0$.

$$F(x) = \sum_{0}^{\infty} \frac{(3x)^{n+1}}{(n+1)^2} + C = \frac{3x}{(1)^2} + \frac{(3x)^2}{(2)^2} + \frac{(3x)^3}{(3)^2} + \cdots + C$$

$F(0) = 0 \Rightarrow \underline{C = 0}$.

(17) n^{th} term of TS for $f(x) = \frac{1}{(1+x)^2}$; $c = 0$

$f(x) = (1+x)^{-2}$ $\qquad f(0) = 1 = 1$

$f^{(1)}(x) = -2(1+x)^{-3}$ $\qquad f^{(1)}(0) = -2 = -2!$

$f^{(2)} = 3 \cdot 2 (1+x)^{-4}$ $\qquad f^{(2)}(0) = 3 \cdot 2 = 3!$

$f^{(3)} = 4 \cdot 3 \cdot 2 (1+x)^{-5}$ $\qquad f^{(3)}(0) = 4 \cdot 3 \cdot 2 = 4!$

$f^{(4)} = -5 \cdot 4 \cdot 3 \cdot 2 (1+x)^{-6}$ $\qquad f^{(4)}(0) = -5 \cdot 4 \cdot 3 \cdot 2 = -5!$

$f^{(n)} = (-1)^n (n+1)! (1+x)^{-(n+2)}$ $\qquad f^{(n)}(0) = (-1)^n (n+1)!$

$\underline{n^{th} \text{ term}}$ $\qquad \frac{f^{(n)}(0)}{n!} x^n = \frac{(-1)^n (n+1)!}{n!} x^n$

$$= \boxed{(-1)^n (n+1) x^n}$$