Math 21 - Review Exam 1

Part One

① $x = \frac{1}{t}$ $y = 2t^3 + 4$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{6t^2}{-\frac{1}{t^2}} = -6t^4$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(-6t^4)}{x'(t)} = \frac{-24t^3}{-\frac{1}{t^2}} = +24t^5$$

② $x = \sec\theta$ $y = \cos^2\theta$

tan. line at $\theta = 0$

$$\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)} = \frac{2\cos\theta(-\sin\theta)}{\sec\theta\tan\theta} =$$

$$= \frac{-2\cos\theta\,\cancel{\sin\theta}}{\frac{1}{\cos\theta}\cdot\frac{\cancel{\sin\theta}}{\cos\theta}} = -2\cos^3\theta.$$

$$\left.\frac{dy}{dx}\right|_{\theta=0} = -2(\overset{1}{\cancel{\cos}\,0})^3 = -2.$$

For tangent line: slope $= -2$

point $(x, y) = (x(0), y(0)) = (\overset{1}{\cancel{\sec}}(0), \overset{1}{\cancel{\cos^2}}(0))$

$= (1, 1)$. Tan. line: $(y-1) = -2(x-1)$.

③ Tan. line to $x = \frac{1}{2t+1}$, $y = 3t-2$ at $t=1$.

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{3}{\frac{-2}{(2t+1)^2}} = -\frac{3}{2}(2t+1)^2$$

$$\left.\frac{dy}{dx}\right|_{t=1} = -\frac{3}{2}(3)^2 = \frac{-3}{2}(9) = \frac{-27}{2}.$$

Point: when $t=1$, $(x,y) = (\frac{1}{3}, 1)$.

tan line: $y - 1 = -\frac{27}{2}(x - \frac{1}{3})$.

④ $x^2 + y^2 - 3x + 2y = 0$ ← polar?

$$\boxed{\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ x^2+y^2 = r^2 \end{array}}$$

$r^2 - 3r\cos\theta + 2r\sin\theta = 0$

$r(r - 3\cos\theta + 2\sin\theta = 0)$

$r - 3\cos\theta + 2\sin\theta = 0$

$r = 3\cos\theta + 2\sin\theta$.

⑤ $r = \frac{3}{5\sin\theta - 2\cos\theta}$ ← rectangular?

$$\frac{r}{r} = \frac{3}{5r\sin\theta - 2r\cos\theta}$$

$$1 = \frac{3}{5y - 2x}$$

⑥ A) arc length of $x = t - \frac{t^3}{9}$

$y = \frac{t^2}{...} + 1$, $0 \le t \le 1$

$$s = \int_0^1 \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

$$= \int_0^1 \sqrt{\left(1 - \frac{1}{3}t^2\right)^2 + \left(\frac{2}{\sqrt{3}}t\right)^2}\, dt$$

$$= \int_0^1 \sqrt{1 - \frac{2}{3}t^2 + \frac{1}{9}t^4 + \frac{4}{3}t^2}\, dt$$

$$= \int_0^1 \sqrt{\frac{1}{9}t^4 + \frac{2}{3}t^2 + 1}\, dt$$

$$= \int_0^1 \sqrt{\left(\frac{1}{3}t^2 + 1\right)^2}\, dt = \int_0^1 \frac{t^2}{3} + 1\, dt$$

$$= \left[\frac{1}{3}\frac{t^3}{3} + t\right]_0^1 = \frac{1}{9} + 1 = \boxed{\frac{10}{9}}$$

B) arc length $x = 1 - \sin t$, $y = 1 - \cos t$,

for $0 \le t \le 2\pi$.

$$s = \int_0^{2\pi} \sqrt{(-\cos t)^2 + (\sin t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{\underbrace{\cos^2 t + \sin^2 t}_{1}}\, dt = [t]_0^{2\pi} = \boxed{2\pi}.$$

c) arc length $x = t^2$, $y = 4t^3$,

$0 \le t \le 1$.

$$s = \int_0^1 \sqrt{(2t)^2 + (12t^2)^2}\, dt$$

$$= \int_0^1 \sqrt{4t^2+144t^4}\,dt = \int_0^1 \sqrt{4t^2(1+36t^2)}\,dt$$

$$= \int_0^1 2t\sqrt{1+36t^2}\,dt = \left[\cancel{2}\cdot\frac{1}{\cancel{72}\,36}(1+36t^2)^{\frac{3}{2}}\cdot\frac{2}{3}\right]_0^1$$

$$= \left[\frac{1}{54}(1+36t^2)^{3/2}\right]_0^1 =$$

$$\frac{1}{54}(37)^{3/2} - \frac{1}{54}(1) .$$

Part Two

① $\bar{u} = \langle 3,-2,0\rangle$, $\bar{v} = \langle -1,2,-3\rangle$

A) $\bar{u}\cdot\bar{v} = -3-4+0 = \boxed{-7}$

B) $(2\bar{u}-3\bar{v})\cdot\bar{v} =$

$(\langle 6,-4,0\rangle - \langle -3,6,-9\rangle)\cdot\bar{v}$

$= \langle 9,-10,9\rangle\cdot\langle -1,2,-3\rangle$

$= -9-20-27 = \boxed{-56}$

C) $\text{proj}_{\bar{u}}\bar{v} = \dfrac{\bar{u}\cdot\bar{v}}{\|\bar{u}\|^2}\cdot\bar{u}$

$$= \frac{-7}{9+4+0}\langle 3,-2,0\rangle = \frac{-7}{13}\langle 3,-2,0\rangle$$

D) $\bar{u} = \langle 3,-2,0\rangle$
$\bar{v} = \langle -1,2,-3\rangle$

$$\bar{u}\times\bar{v} = \begin{vmatrix} -2 & 0 \\ 2 & -3 \end{vmatrix}\bar{i} - \begin{vmatrix} 3 & 0 \\ -1 & -3 \end{vmatrix}\bar{j} + \begin{vmatrix} 3 & -2 \\ -1 & 2 \end{vmatrix}\bar{k}$$

$= (6-0)\bar{i} - (-9-0)\bar{j} + (6-2)\bar{k}$
$= 6\bar{i} + 9\bar{j} + 4\bar{k}$

E) $\bar{u} \cdot \bar{v} = \|\bar{u}\| \|\bar{v}\| \cos\theta$

$-7 = \sqrt{9+4}\sqrt{1+4+9} \cos\theta$

$-.519 = \dfrac{-7}{\sqrt{13}\sqrt{14}} = \cos\theta$

$\theta = \boxed{121^{0}}$

F) $\|\bar{u}\| = \sqrt{9+4+0} = \sqrt{13}$

(3) Area of Δ with verticies

$P(2,-1,3)$, $Q(3,0,2)$, $R(0,2,1)$

$\bar{u} = \overrightarrow{PQ} = \langle 1, 1, -1 \rangle$

$\bar{v} = \overrightarrow{PR} = \langle -2, 3, -2 \rangle$

area $= \frac{1}{2} \|\bar{u} \times \bar{v}\|$.

$\bar{u} \times \bar{v} = \begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix} \bar{i} - \begin{vmatrix} 1 & -1 \\ -2 & -2 \end{vmatrix} \bar{j} + \begin{vmatrix} 1 & 1 \\ -2 & 3 \end{vmatrix} \bar{k}$

$= (-2-(-3))\bar{i} - (-2-2)\bar{j} + (3-(-2))\bar{k}$
$= (1)\bar{i} + 4\bar{j} + 5\bar{k}$

area $= \frac{1}{2}\|\bar{u} \times \bar{v}\| = \frac{1}{2}\sqrt{1+16+25}$
$= \frac{1}{2}\sqrt{42}$.

(3) Line thru $P(1,-2,3)$ and $Q(0,2,5)$

direction vector $\bar{u} = \langle -1, 4, 2 \rangle$

line : $x = 1 + (-1)t$

$y = -2 + (4)t$

$z = 3 + (2)t$

(4) $P(0,2,-1)$, $Q(1,0,3)$, $R(2,-1,1)$

Eq plane ?

$\bar{u} = \overrightarrow{PQ} = \langle 1, -2, 4 \rangle$

$\bar{v} = \overrightarrow{PR} = \langle 2, -3, 2 \rangle$

$\bar{n} = \bar{u} \times \bar{v} = \begin{vmatrix} -2 & 4 \\ -3 & 2 \end{vmatrix} \bar{i} - \begin{vmatrix} 1 & 4 \\ 2 & 2 \end{vmatrix} \bar{j} + \begin{vmatrix} 1 & -2 \\ 2 & -3 \end{vmatrix} \bar{k}$

$= (-4 + 12)\bar{i} - (2 - 8)\bar{j} + (-3 + 4)\bar{k}$

$= 8\bar{i} + 6\bar{j} + \bar{k}$

Equa of plane with direc vector $\langle 8, 6, 1 \rangle$
and thru pt $(0, 2, -1)$:

$$\boxed{8(x-0) + 6(y-2) + 1(z+1) = 0}$$

(5) Dist from point $P(2,3,-1)$
to plane $3x - y + 2z - 2 = 0$.

$$\text{dist} = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|(3)(2) + (-1)(3) + (2)(-1) - 2|}{\sqrt{(3)^2 + (-1)^2 + (2)^2}}$$

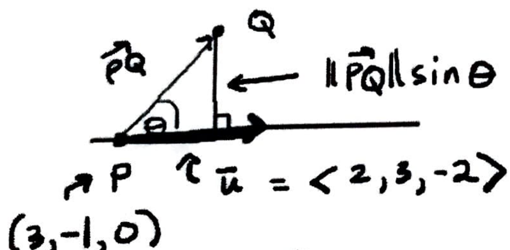
$$= \frac{|6 - 3 - 2 - 2|}{\sqrt{9 + 1 + 4}} = \boxed{\frac{1}{\sqrt{14}}}$$

⑥ Dist between $Q(1,2,-1)$ and line

$x = 3 + 2t$

$y = -1 + 3t$

$z = 0 - 2t$



$\|\bar{u} \times \bar{v}\| = \|\bar{u}\| \, \|\bar{v}\| \sin\theta$

$$\Rightarrow \quad \|\vec{PQ}\| \sin\theta = \frac{\|\bar{u} \times \vec{PQ}\|}{\|\bar{u}\|}$$

$\bar{u} = \langle 2, 3, -2 \rangle$

$\vec{PQ} = \langle -2, 3, -1 \rangle$

$$\bar{u} \times \vec{PQ} = \begin{vmatrix} 3 & -2 \\ 3 & -1 \end{vmatrix} \bar{i} - \begin{vmatrix} 2 & -2 \\ -2 & -1 \end{vmatrix} \bar{j} + \begin{vmatrix} 2 & 3 \\ -2 & 3 \end{vmatrix} \bar{k}$$

$$= (-3 + 6)\bar{i} - (-2 - 4)\bar{j} + (6 + 6)\bar{k}$$

$$= 3\bar{i} + 6\bar{j} + 12\bar{k} = \langle 3, 6, 12 \rangle$$

$$\frac{\|\langle 3, 6, 12 \rangle\|}{\|\langle 2, 3, -2 \rangle\|} = \frac{\sqrt{9 + 36 + 144}}{\sqrt{4 + 9 + 4}} = \frac{\sqrt{189}}{\sqrt{17}}$$

7 $\measuredangle$ between planes

$2x-3y+2z=0$ and $3x-y+z-1=0$

$\vec{n}_1=\langle 2,-3,2\rangle$ $\quad$ $\vec{n}_2=\langle 3,-1,1\rangle$

$\vec{n}_1\cdot\vec{n}_2=\|\vec{n}_1\|\,\|\vec{n}_2\|\cos\theta$

$6+3+2=\sqrt{4+9+4}\,\sqrt{9+1+1}\cos\theta$

$$\frac{11}{\sqrt{17}\,\sqrt{11}}=\cos\theta$$

$.804=\cos\theta$

$\boxed{36^\circ}=\theta$

<u>Part three</u>

1 $$\frac{x^2}{4}+\frac{y^2}{9}-\frac{z^2}{16}=-1$$

trace:

A) xy-plane $\quad \frac{x^2}{4}+\frac{y^2}{9}=-1 \Rightarrow$ no trace

B) xz-plane $\quad \frac{x^2}{4}-\frac{z^2}{16}=-1$

$$\frac{z^2}{16}-\frac{x^2}{4}=1$$

hyperbola, axis is z-axis

c) yz-plane $\frac{y^2}{9} - \frac{z^2}{16} = -1$

$$\frac{z^2}{16} - \frac{y^2}{9} = 1$$

hyperbola, axis is z-axis.

D) plane $z = 4$

$$\frac{x^2}{4} + \frac{y^2}{9} - 1 = -1$$

$$\frac{x^2}{4} + \frac{y^2}{9} = 0$$ ← one point $(0,0,4)$

E) plane $z = 2$

$$\frac{x^2}{4} + \frac{y^2}{9} - \frac{4}{16} = -1$$

$$\frac{x^2}{4} + \frac{y^2}{9} = -1 + \frac{1}{4}$$

$$\frac{x^2}{4} + \frac{y^2}{9} = \frac{-3}{4}$$ ← no trace

F) in plane $z=5$

$$\frac{x^2}{4}+\frac{y^2}{9}-\frac{25}{16}=-1$$

$$\frac{x^2}{4}+\frac{y^2}{9}=\frac{25}{16}-\frac{16}{16}$$

$$\frac{x^2}{4}+\frac{y^2}{9}=\frac{9}{16}$$ } ellipse

② $x^2+z^2=4$ ← 3D-cyliner (rulings along y)

A) trace in $y=4$ is $x^2+z^2=4$

B) rulings ∥ to y-axis