took up the challenge, and the following year the problem was solved by Newton, Leibniz, L'Hôpital, John Bernoulli, and James Bernoulli. As it turns out, the solution is not a straight line from A to B, but an inverted cycloid passing through the points A and B, as shown in Figure 12.8. The amazing part of the solution is that a particle starting at rest at *any* other point C of the cycloid between A and B will take exactly the same time to reach B.

James Bernoulli (1654–1705), also called Jacques, was the older brother of John. He was one of several accomplished mathematicians of the Swiss Bernoulli family. James's mathematical accomplishments have given him a prominent place in the early development of calculus.

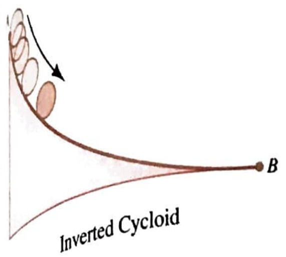


FIGURE 12.8

EXERCISES for Section 12.1

1. Consider the parametric equations $x = \sqrt{t}$ and $y = 1 - t$.

(a) Complete the table.

t	0	1	2	3	4
x					
y					

(b) Plot the points (x, y) generated in the table and sketch a graph of the parametric equations. Indicate the orientation of the graph.

(c) Find the rectangular equation by eliminating the parameter. Compare the graph of part (b) with the graph of the rectangular equation.

2. Consider the parametric equations $x = 4\cos^2\theta$ and $y = 2\sin\theta$.

(a) Complete the table.

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
x					
y					

(b) Plot the points (x, y) generated in the table and sketch a graph of the parametric equations. Indicate the orientation of the graph.

(c) Find the rectangular equation by eliminating the parameter. Compare the graph of part (b) with the graph of the rectangular equation.

(d) If values of θ were selected from the interval $[\pi/2, 3\pi/2]$ for the table in part (a), would the graph of part (b) be different? Why or why not?

In Exercises 3–30, sketch the curve represented by the parametric equations (indicate the orientation of the curve), and find the corresponding rectangular equation by eliminating the parameter.

3. $x = 3t - 1$, $y = 2t + 1$

4. $x = 3 - 2t$, $y = 2 + 3t$

5. $x = \sqrt[3]{t}$, $y = 1 - t$

6. $x = t + 1$, $y = t^2$

7. $x = t + 1$, $y = t^3$

8. $x = t^3$, $y = \dfrac{t^2}{2}$

9. $x = 1 + \dfrac{1}{t}$, $y = t - 1$

10. $x = t - 1$, $y = \dfrac{t}{t - 1}$

11. $x = t^2 + t$, $y = t^2 - t$

12. $x = 2t$, $y = |t - 2|$

13. $x = |t - 1|$, $y = t + 2$

14. $x = \sec\theta$, $y = \cos\theta$

15. $x = \tan^2\theta$, $y = \sec^2\theta$

16. $x = 3\cos\theta$, $y = 3\sin\theta$

17. $x = \cos\theta$, $y = 3\sin\theta$

18. $x = 4\sin 2\theta$, $y = 2\cos 2\theta$

19. $x = \cos\theta$, $y = 2\sin 2\theta$

20. $x = \cos\theta$, $y = 2\sin^2\theta$

21. $x = 4 + 2\cos\theta$, $y = -1 + \sin\theta$

22. $x = 4 + 2\cos\theta$, $y = -1 + 2\sin\theta$

23. $x = 4 + 2\cos\theta$, $y = -1 + 4\sin\theta$

24. $x = \sec\theta$, $y = \tan\theta$

25. $x = 4\sec\theta$, $y = 3\tan\theta$

26. $x = \cos^3\theta$, $y = \sin^3\theta$

27. $x = t^3$, $y = 3\ln t$

28. $x = e^{2t}$, $y = e^t$

29. $x = e^{-t}$, $y = e^{3t}$

30. $x = \ln 2t$, $y = t^2$

EXERCISES for Section 12.2

In Exercises 1–10, find dy/dx and d^2y/dx^2, and evaluate each at the specified value of the parameter.

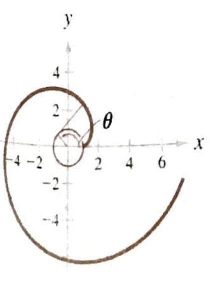
Parametric equations	*Point*
1. $x = 2t,\ y = 3t - 1$	$t = 3$
2. $x = \sqrt{t},\ y = 3t - 1$	$t = 1$
3. $x = t + 1,\ y = t^2 + 3t$	$t = -1$
4. $x = t^2 + 3t,\ y = t + 1$	$t = 0$
5. $x = 2\cos\theta,\ y = 2\sin\theta$	$\theta = \dfrac{\pi}{4}$
6. $x = \cos\theta,\ y = 3\sin\theta$	$\theta = 0$
7. $x = 2 + \sec\theta,\ y = 1 + 2\tan\theta$	$\theta = \dfrac{\pi}{6}$
8. $x = \sqrt{t},\ y = \sqrt{t - 1}$	$t = 2$
9. $x = \cos^3\theta,\ y = \sin^3\theta$	$\theta = \dfrac{\pi}{4}$
10. $x = \theta - \sin\theta,\ y = 1 - \cos\theta$	$\theta = \pi$

In Exercises 11–16, find an equation of the tangent line to the curve at the specified value of the parameter.

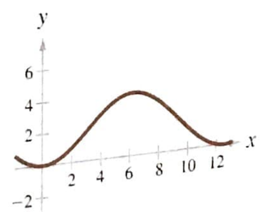
Parametric equations	*Point*
11. $x = 2t,\ y = t^2 - 1$	$t = 2$
12. $x = t - 1,\ y = \dfrac{1}{t} + 1$	$t = 1$
13. $x = t^2 - t + 2,\ y = t^3 - 3t$	$t = -1$
14. $x = 4\cos\theta,\ y = 3\sin\theta$	$\theta = \dfrac{3\pi}{4}$
15. $x = 2\cot\theta,\ y = 2\sin^2\theta$	$\theta = \dfrac{\pi}{4}$
16. $x = 2 - 3\cos\theta,\ y = 3 + 2\sin\theta$	$\theta = \dfrac{5\pi}{3}$

In Exercises 17 and 18, find all points (if any) of horizontal and vertical tangency to the portion of the curve.

17. Involute of a circle:
$x = \cos\theta + \theta\sin\theta$
$y = \sin\theta - \theta\cos\theta$



18. $x = 2\theta$
$y = 2(1 - \cos\theta)$



In Exercises 19–28, find all points (if any) of horizontal and vertical tangency.

19. $x = 1 - t$
$y = t^2$

20. $x = t + 1$
$y = t^2 + 3t$

21. $x = 1 - t$
$y = t^3 - 3t$

22. $x = t^2 - t + 2$
$y = t^3 - 3t$

23. $x = 3\cos\theta$
$y = 3\sin\theta$

24. $x = \cos\theta$
$y = 2\sin 2\theta$

25. $x = 4 + 2\cos\theta$
$y = -1 + \sin\theta$

26. $x = 4\cos^2\theta$
$y = 2\sin\theta$

27. $x = \sec\theta$
$y = \tan\theta$

28. $x = \cos^2\theta$
$y = \cos\theta$

In Exercises 29–34, find the arc length of the given curve.

Parametric equations	*Interval*
29. $x = e^{-t}\cos t,\ y = e^{-t}\sin t$	$0 \le t \le \dfrac{\pi}{2}$
30. $x = t^2,\ y = 4t^3 - 1$	$-1 \le t \le 1$
31. $x = t^2,\ y = 2t$	$0 \le t \le 2$
32. $x = \arcsin t,\ y = \ln\sqrt{1 - t^2}$	$0 \le t \le \dfrac{1}{2}$
33. $x = \sqrt{t},\ y = 3t - 1$	$0 \le t \le 1$
34. $x = t,\ y = \dfrac{t^5}{10} + \dfrac{1}{6t^3}$	$1 \le t \le 2$

In Exercises 35–38, find the arc length of the given curve on the interval $[0, 2\pi]$.

35. Perimeter of a hypocycloid: $x = a\cos^3\theta$
$y = a\sin^3\theta$

36. Circumference of a circle: $x = a\cos\theta$
$y = a\sin\theta$

37. One arch of a cycloid: $x = a(\theta - \sin\theta)$
$y = a(1 - \cos\theta)$

38. Involute of a circle: $x = \cos\theta + \theta\sin\theta$
$y = \sin\theta - \theta\cos\theta$

39. *Folium of Descartes* Given the parametric equations

$$x = \frac{4t}{1 + t^3} \quad \text{and} \quad y = \frac{4t^2}{1 + t^3}$$

use a graphing utility to perform the following.

(a) Sketch the curve described by the parametric equations.

(b) Find the points of horizontal tangency to the curve.

(c) Use Simpson's Rule with $n = 10$ to approximate the arc length of the closed loop (Hint: Because of symmetry we need only integrate over the interval $0 \le t \le 1$.)

EXERCISES for Section 12.3

In Exercises 1–8, the polar coordinates of a point are given. Plot the point and find the corresponding rectangular coordinates.

1. $\left(4, \frac{3\pi}{6}\right)$
2. $\left(4, \frac{3\pi}{2}\right)$
3. $\left(-1, \frac{5\pi}{4}\right)$
4. $(0, -\pi)$
5. $\left(4, -\frac{\pi}{3}\right)$
6. $\left(-1, -\frac{3\pi}{4}\right)$
7. $(\sqrt{2}, 2.36)$
8. $(-3, -1.57)$

In Exercises 9–16, the rectangular coordinates of a point are given. Find two sets of polar coordinates for the point, using $0 \leq \theta < 2\pi$.

9. $(1, 1)$
10. $(0, -5)$
11. $(-3, 4)$
12. $(3, -1)$
13. $(-\sqrt{3}, -\sqrt{3})$
14. $(-2, 0)$
15. $(4, 6)$
16. $(5, 12)$

In Exercises 17–32, convert the given rectangular equation to polar form.

17. $x^2 + y^2 = 9$
18. $x^2 + y^2 = a^2$
19. $x^2 + y^2 - 2ax = 0$
20. $x^2 + y^2 - 2ay = 0$
21. $y = 4$
22. $y = b$
23. $x = 10$
24. $x = a$
25. $3x - y + 2 = 0$
26. $4x + 7y - 2 = 0$
27. $xy = 4$
28. $y = x$
29. $y^2 = 9x$
30. $y^2 - 8x - 16 = 0$
31. $x^2 - 4ay - 4a^2 = 0$
32. $(x^2 + y^2)^2 - 9(x^2 - y^2) = 0$

In Exercises 33–44, convert the given polar equation to rectangular form.

33. $r = 4 \sin \theta$
34. $r = 4 \cos \theta$
35. $\theta = \frac{\pi}{6}$
36. $r = 4$
37. $r = 2 \csc \theta$
38. $r^2 = \sin 2\theta$
39. $r = 1 - 2 \sin \theta$
40. $r = \frac{1}{1 - \cos \theta}$
41. $r = \frac{6}{2 - 3 \sin \theta}$
42. $r = 2 \sin 3\theta$
43. $r^2 = 4 \sin \theta$
44. $r = \frac{6}{2 \cos \theta - 3 \sin \theta}$

In Exercises 45–54, sketch the graph of the polar equation. Indicate any symmetry possessed by the graph.

45. $r = 5$
46. $r = -2$
47. $r = \theta$
48. $r = \frac{\theta}{\pi}$
49. $r = \sin \theta$
50. $r = 3 \cos \theta$
51. $r = 2 \sec \theta$
52. $r = 3 \csc \theta$
53. $r = 4(2 + \sin \theta)$
54. $r = 2 + \cos \theta$

55. Convert the equation

$$r = 2(h \cos \theta + k \sin \theta)$$

to rectangular form and verify that it is the equation of a circle. Find the radius and the rectangular coordinates of the center of the circle.

56. Verify that the distance between the two points (r_1, θ_1) and (r_2, θ_2) in polar coordinates is given by

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_1 - \theta_2)}.$$

12.4 Tangent Lines and Curve Sketching in Polar Coordinates

Relative extrema of r ▪ Tangent lines to polar graphs ▪ Special polar graphs

We have seen that polar curve sketching can be simplified by considering the periodic nature and symmetry of the graph. We begin this section by using calculus to develop two additional curve sketching aids.

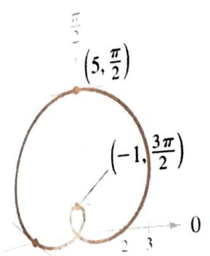
If r is a differentiable function of θ, then we can determine the **relative extrema of** r using the procedures discussed in Chapter 4. However, since the distance between a point (r, θ) and the pole is $|r|$, it is possible that both the relative maxima and relative minima of r yield points whose distance from the pole is a relative maximum. We illustrate this with an example.

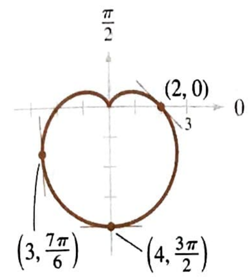
EXERCISES for Section 12.4

In Exercises 1 and 2, find dy/dx and the slope of the tangent lines shown on the graph of the polar equation.

1. $r = 2 + 3 \sin \theta$

2. $r = 2(1 - \sin \theta)$





In Exercises 3–6, find dy/dx and the slope of the graph of the polar curve at the given value of θ.

Polar Equation	Value of θ
3. $r = 3(1 - \cos \theta)$	$\theta = \frac{\pi}{2}$
4. $r = 3 - 2 \cos \theta$	$\theta = 0$
5. $r = 3 \sin \theta$	$\theta = \frac{\pi}{3}$
6. $r = 4$	$\theta = \frac{\pi}{4}$

In Exercises 7 and 8, find the points of horizontal and vertical tangency (if any) to the polar curve.

7. $r = 1 + \sin \theta$
8. $r = a \sin \theta$

In Exercises 9 and 10, find the points of horizontal tangency (if any) to the polar curve.

9. $r = 2 \csc \theta + 3$
10. $r = a \sin \theta \cos^2 \theta$

In Exercises 11–14, use a graphics utility to graph the polar equation and find all points of horizontal tangency.

11. $r = 4 \sin \theta \cos^2 \theta$
12. $r = 3 \cos 2\theta \sec \theta$
13. $r = 2 \csc \theta + 5$
14. $r = 2 \cos(3\theta - 2)$

In Exercises 15–22, sketch the graph of the polar equation and find the tangents at the pole.

15. $r = 3 \sin \theta$
16. $r = 3(1 - \cos \theta)$
17. $r = 2 \cos 3\theta$
18. $r = -\sin 5\theta$
19. $r = 3 \sin 2\theta$
20. $r = 3 \cos 2\theta$
21. $r^2 = 4 \cos 2\theta$
22. $r^2 = 4 \sin \theta$

In Exercises 23–28, sketch the graph of the polar equation.

23. $r = 3 - 2 \cos \theta$
24. $r = 5 - 4 \sin \theta$
25. $r = 3 \csc \theta$
26. $r = \dfrac{6}{2 \sin \theta - 3 \cos \theta}$
27. $r = 2\theta$
28. $r = \dfrac{1}{\theta}$

In Exercises 29–36, use a graphing utility to graph the polar equation. Find an interval for θ over which the graph is traced *only once*.

29. $r = 3 - 4 \cos \theta$
30. $r = 2(1 - 2 \sin \theta)$
31. $r = 2 + \sin \theta$
32. $r = 4 + 3 \cos \theta$
33. $r = 2 \cos \left(\dfrac{3\theta}{2}\right)$
34. $r = 3 \sin \left(\dfrac{5\theta}{2}\right)$
35. $r^2 = 4 \sin 2\theta$
36. $r^2 = 3 \cos 4\theta$

In Exercises 37–40, use a graphing utility to graph the equation and show that the indicated line is an asymptote to the graph.

Name of Graph	Polar Equation	Asymptote
37. Conchoid	$r = 2 - \sec \theta$	$x = -1$
38. Conchoid	$r = 2 + \csc \theta$	$y = 1$
39. Hyperbolic spiral	$r = \dfrac{2}{\theta}$	$y = 2$
40. Strophoid	$r = 2 \cos 2\theta \sec \theta$	$x = -2$

41. Sketch the graph of $r = 4 \sin \theta$ over each of the following intervals.
(a) $0 \le \theta \le \frac{\pi}{2}$
(b) $\frac{\pi}{2} \le \theta \le \pi$
(c) $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
(d) $\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$

42. Use a graphing utility to graph the polar equation
$$r = 6[1 + \cos(\theta - \phi)]$$
for (a) $\phi = 0$, (b) $\phi = \pi/4$, and (c) $\phi = \pi/2$. Use these graphs to describe the effect of the angle ϕ. Write the equation as a function of $\sin \theta$ for part (c).

43. Verify that if the curve whose polar equation is $r = f(\theta)$ is rotated about the pole through an angle ϕ, then an equation for the rotated curve is $r = f(\theta - \phi)$.

44. If the polar form of an equation for a curve is $r = f(\sin \theta)$, show that the form becomes
(a) $r = f(-\cos \theta)$ if the curve is rotated counterclockwise $\pi/2$ radians about the pole.

EXERCISES for Section 12.5

In Exercises 1 and 2, find the area of the region bounded by the graph of the polar equation by (a) geometric formula and (b) by integration.

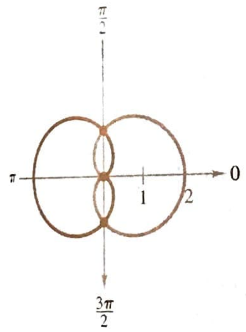
1. $r = 8 \sin \theta$
2. $r = 3 \cos \theta$

In Exercises 3–12, find the area of the region.

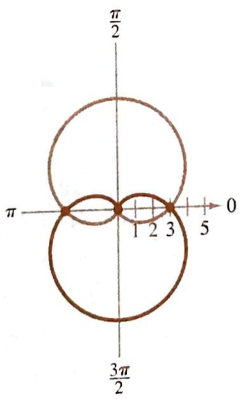
3. One petal of $r = 2 \cos 3\theta$.
4. One petal of $r = 4 \sin 2\theta$
5. One petal of $r = \cos 2\theta$
6. One petal of $r = \cos 5\theta$
7. Interior of $r = 1 - \sin \theta$
8. Interior of $r = 1 - \sin \theta$ (above the polar axis)
9. Inner loop of $r = 1 + 2 \cos \theta$
10. Inner loop of $r = 3 + 4 \sin \theta$
11. Between the loops of $r = 1 + 2 \cos \theta$
12. Between the loops of $r = 2(1 + 2 \sin \theta)$

In Exercises 13–22, find the points of intersection of the graphs of the equations.

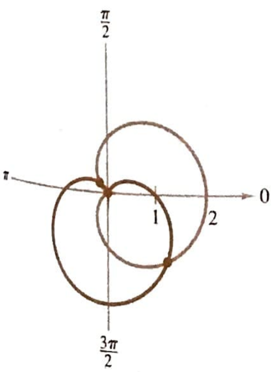
13. $r = 1 + \cos \theta$
$r = 1 - \cos \theta$



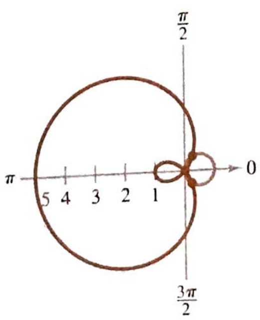
14. $r = 3(1 + \sin \theta)$
$r = 3(1 - \sin \theta)$



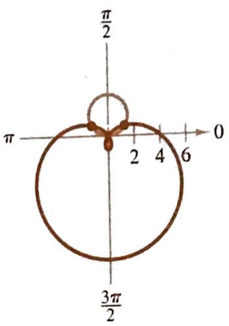
15. $r = 1 + \cos \theta$
$r = 1 - \sin \theta$



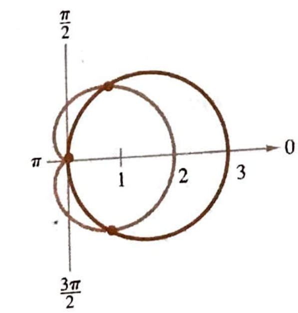
16. $r = 2 - 3 \cos \theta$
$r = \cos \theta$



17. $r = 4 - 5 \sin \theta$
$r = 3 \sin \theta$



18. $r = 1 + \cos \theta$
$r = 3 \cos \theta$



19. $r = \dfrac{\theta}{2}$, $r = 2$
20. $\theta = \dfrac{\pi}{4}$, $r = 2$
21. $r = 4 \sin 2\theta$, $r = 2$
22. $r = 3 + \sin \theta$, $r = 2 \csc \theta$

In Exercises 23 and 24, use a graphing utility to find the points of intersection of the graphs of the polar equations. Watch the graphs as they are traced on the display screen. Explain why the pole is not a point of intersection obtained by solving the equations simultaneously.

23. $r = \cos \theta$
$r = 2 - 3 \sin \theta$
24. $r = 4 \sin \theta$
$r = 2(1 + \sin \theta)$

In Exercises 25–32, find the area of the region.

25. Common interior of $r = 4 \sin 2\theta$ and $r = 2$
26. Common interior of $r = 3(1 + \sin \theta)$ and $r = 3(1 - \sin \theta)$
27. Common interior of $r = 3 - 2 \sin \theta$ and $r = -3 + 2 \sin \theta$
28. Common interior of $r = 3 - 2 \sin \theta$ and $r = 3 - 2 \cos \theta$
29. Common interior of $r = 4 \sin \theta$ and $r = 2$
30. Inside $r = 3 \sin \theta$ and outside $r = 2 - \sin \theta$
31. Inside $r = a(1 + \cos \theta)$ and outside $r = a \cos \theta$
32. Inside $r = 2a \cos \theta$ and outside $r = a$

33. The radiation from a transmitting antenna is not uniform in all directions. The intensity from a particular antenna is modeled by

$$r = a \cos^2 \theta.$$

(a) Convert the polar equation to rectangular form.
(b) Use a graphing utility to graph the model for $a = 4$ and $a = 6$.
(c) Find the area of the geographical region between the two curves of part (b).

EXAMPLE 7 An application involving velocity

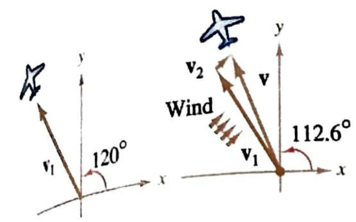


FIGURE 13.12

An airplane is traveling at a fixed altitude with a negligible wind factor. The plane is headed N 30° W at a speed of 500 miles per hour, as shown in Figure 13.12. As the plane reaches a certain point, it encounters wind with a velocity of 70 miles per hour in the direction E 45° N. What is the resultant speed and direction of the plane?

SOLUTION

Using Figure 13.12, we can represent the velocity of the plane by the vector

$$\mathbf{v}_1 = 500\langle\cos(120°), \sin(120°)\rangle = 500\cos(120°)\,\mathbf{i} + 500\sin(120°)\,\mathbf{j}.$$

The velocity of the wind is represented by the vector

$$\mathbf{v}_2 = 70\langle\cos(45°), \sin(45°)\rangle = 70\cos(45°)\,\mathbf{i} + 70\sin(45°)\,\mathbf{j}.$$

The resultant velocity of the plane is

$$\begin{aligned}\mathbf{v} &= \mathbf{v}_1 + \mathbf{v}_2\\ &= 500\cos(120°)\,\mathbf{i} + 500\sin(120°)\,\mathbf{j} + 70\cos(45°)\,\mathbf{i} + 70\sin(45°)\,\mathbf{j}\\ &= [500\cos(120°) + 70\cos(45°)]\,\mathbf{i} + [500\sin(120°) + 70\sin(45°)]\,\mathbf{j}\\ &\approx -200.5\mathbf{i} + 482.5\mathbf{j}.\end{aligned}$$

Now, to find the speed and direction, we write $\mathbf{v} = \|\mathbf{v}\|(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})$. Since

$$\|\mathbf{v}\| \approx \sqrt{(-200.5)^2 + (482.5)^2} \approx 522.5$$

we have

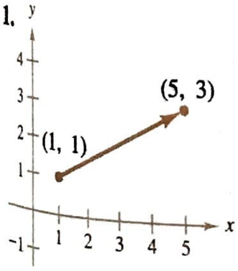
$$\begin{aligned}\mathbf{v} &\approx 522.5\left[\frac{-200.5}{522.5}\mathbf{i} + \frac{482.5}{522.5}\mathbf{j}\right] \approx 522.5[-0.3837\mathbf{i} + 0.9234\mathbf{j}]\\ &\approx 522.5[\cos(112.6°)\,\mathbf{i} + \sin(112.6°)\,\mathbf{j}].\end{aligned}$$

Therefore, the speed of the plane, as altered by the wind, is approximately 522.5 miles per hour in a flight path that makes an angle of 112.6° with the positive x-axis. ▭

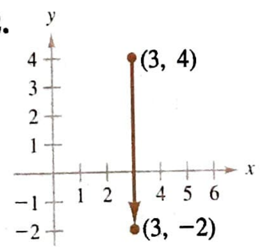
EXERCISES for Section 13.1

In Exercises 1–4, (a) find the component form of the vector $\mathbf{v}$ and (b) sketch the vector with its initial point at the origin.

1.



2.



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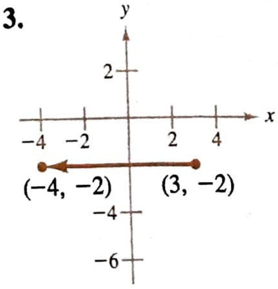

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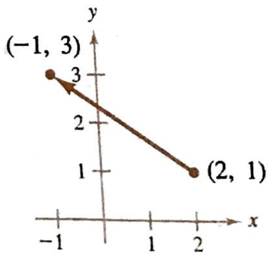

In Exercises 5–12, the initial and terminal points of a vector $\mathbf{v}$ are given. (a) Sketch the given directed line segment, (b) write the vector in component form, and (c) sketch the vector with its initial point at the origin.

	Initial Point	*Terminal Point*
5.	$(1, 2)$	$(5, 5)$
6.	$(3, -5)$	$(4, 7)$
7.	$(10, 2)$	$(6, -1)$
8.	$(0, -4)$	$(-5, -1)$
9.	$(6, 2)$	$(6, 6)$
10.	$(7, -1)$	$(-3, -1)$
11.	$\left(\frac{3}{2}, \frac{4}{3}\right)$	$\left(\frac{1}{2}, 3\right)$
12.	$(0.12, 0.60)$	$(0.84, 1.25)$

In Exercises 13 and 14, sketch the scalar multiple of $\mathbf{v}$.

13. $\mathbf{v} = \langle 2, 3\rangle$
(a) $2\mathbf{v}$ (b) $-3\mathbf{v}$
(c) $\frac{7}{2}\mathbf{v}$ (d) $\frac{2}{3}\mathbf{v}$

14. $\mathbf{v} = \langle -1, 5\rangle$
(a) $4\mathbf{v}$ (b) $-\frac{1}{2}\mathbf{v}$
(c) $0\mathbf{v}$ (d) $-6\mathbf{v}$

In Exercises 15–20, find the vector $\mathbf{v}$ and illustrate the indicated vector operations geometrically, where $\mathbf{u} = \langle 2, -1\rangle$ and $\mathbf{w} = \langle 1, 2\rangle$.

15. $\mathbf{v} = \frac{3}{2}\mathbf{u}$ **16.** $\mathbf{v} = \mathbf{u} + \mathbf{w}$
17. $\mathbf{v} = \mathbf{u} + 2\mathbf{w}$ **18.** $\mathbf{v} = -\mathbf{u} + \mathbf{w}$
19. $\mathbf{v} = \frac{1}{2}(3\mathbf{u} + \mathbf{w})$ **20.** $\mathbf{v} = \mathbf{u} - 2\mathbf{w}$

In Exercises 21–26, find a and b such that $\mathbf{v} = a\mathbf{u} + b\mathbf{w}$, where $\mathbf{u} = \langle 1, 2\rangle$ and $\mathbf{w} = \langle 1, -1\rangle$.

21. $\mathbf{v} = \langle 2, 1\rangle$ **22.** $\mathbf{v} = \langle 0, 3\rangle$
23. $\mathbf{v} = \langle 3, 0\rangle$ **24.** $\mathbf{v} = \langle 3, 3\rangle$
25. $\mathbf{v} = \langle 1, 1\rangle$ **26.** $\mathbf{v} = \langle -1, 7\rangle$

In Exercises 27 and 28, the vector $\mathbf{v}$ and its initial point are given. Find the terminal point.

27. $\mathbf{v} = \langle -1, 3\rangle$, initial point $(4, 2)$
28. $\mathbf{v} = \langle 4, -9\rangle$, initial point $(3, 2)$

In Exercises 29–34, find the magnitude of $\mathbf{v}$.

29. $\mathbf{v} = \langle 4, 3\rangle$ **30.** $\mathbf{v} = \langle 12, -5\rangle$
31. $\mathbf{v} = 6\mathbf{i} - 5\mathbf{j}$ **32.** $\mathbf{v} = -10\mathbf{i} + 3\mathbf{j}$
33. $\mathbf{v} = 4\mathbf{j}$ **34.** $\mathbf{v} = \mathbf{i} - \mathbf{j}$

In Exercises 35 and 36, find the following.

(a) $\|\mathbf{u}\|$ (b) $\|\mathbf{v}\|$ (c) $\|\mathbf{u} + \mathbf{v}\|$
(d) $\left\|\frac{\mathbf{u}}{\|\mathbf{u}\|}\right\|$ (e) $\left\|\frac{\mathbf{v}}{\|\mathbf{v}\|}\right\|$ (f) $\left\|\frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|}\right\|$

35. $\mathbf{u} = \left\langle 1, \frac{1}{2}\right\rangle$, $\mathbf{v} = \langle 2, 3\rangle$
36. $\mathbf{u} = \langle 2, -4\rangle$, $\mathbf{v} = \langle 5, 5\rangle$

In Exercises 37 and 38, demonstrate the triangle inequality using the vectors $\mathbf{u}$ and $\mathbf{v}$.

37. $\mathbf{u} = \langle 2, 1\rangle$, $\mathbf{v} = \langle 5, 4\rangle$
38. $\mathbf{u} = \langle -3, 2\rangle$, $\mathbf{v} = \langle 1, -2\rangle$

In Exercises 39–42, find the vector $\mathbf{v}$ with the given magnitude and the same direction as $\mathbf{u}$.

	Magnitude	*Direction*
39.	$\|\mathbf{v}\| = 4$	$\mathbf{u} = \langle 1, 1\rangle$
40.	$\|\mathbf{v}\| = 4$	$\mathbf{u} = \langle -1, 1\rangle$
41.	$\|\mathbf{v}\| = 2$	$\mathbf{u} = \langle \sqrt{3}, 3\rangle$
42.	$\|\mathbf{v}\| = 3$	$\mathbf{u} = \langle 0, 3\rangle$

In Exercises 43–46, find a unit vector (a) parallel to and (b) normal to the graph of $f(x)$ at the indicated point.

	Graph	*Point*
43.	$f(x) = x^3$	$(1, 1)$
44.	$f(x) = x^3$	$(-2, -8)$
45.	$f(x) = \sqrt{25 - x^2}$	$(3, 4)$
46.	$f(x) = \tan x$	$\left(\frac{\pi}{4}, 1\right)$

In Exercises 47–50, find the component form of $\mathbf{v}$ given its magnitude and the angle it makes with the positive x-axis.

	Magnitude	*Angle*
47.	$\|\mathbf{v}\| = 3$	$\theta = 0°$
48.	$\|\mathbf{v}\| = 1$	$\theta = 45°$
49.	$\|\mathbf{v}\| = 2$	$\theta = 150°$
50.	$\|\mathbf{v}\| = 1$	$\theta = 3.5°$

In Exercises 51–54, find the component form of $\mathbf{u} + \mathbf{v}$ given the magnitudes of $\mathbf{u}$ and $\mathbf{v}$ and the angles $\mathbf{u}$ and $\mathbf{v}$ make with the positive x-axis.

51. $\|\mathbf{u}\| = 1$, $\theta = 0°$; $\|\mathbf{v}\| = 3$, $\theta = 45°$
52. $\|\mathbf{u}\| = 4$, $\theta = 0°$; $\|\mathbf{v}\| = 2$, $\theta = 60°$
53. $\|\mathbf{u}\| = 2$, $\theta = 4$; $\|\mathbf{v}\| = 1$, $\theta = 2$
54. $\|\mathbf{u}\| = 5$, $\theta = -0.5$; $\|\mathbf{v}\| = 5$, $\theta = 0.5$

In Exercises 55 and 56, find the component form of $\mathbf{v}$ given the magnitudes of $\mathbf{u}$ and $\mathbf{u} + \mathbf{v}$ and the angles $\mathbf{u}$ and $\mathbf{u} + \mathbf{v}$ make with the positive x-axis.

55. $\|\mathbf{u}\| = 1$, $\theta = 45°$; $\|\mathbf{u} + \mathbf{v}\| = \sqrt{2}$, $\theta = 90°$

56. $\|\mathbf{u}\| = 4$, $\theta = 30°$; $\|\mathbf{u} + \mathbf{v}\| = 6$, $\theta = 120°$

57. A force of 150 pounds in a direction 30° above the horizontal is applied to a bolt. Find the horizontal and vertical components of the force (see figure).

58. A 25-pound weight is suspended from the ceiling by a rope 5 feet long. Determine the magnitude of the horizontal force required to hold the weight 1 foot from the vertical (see figure).

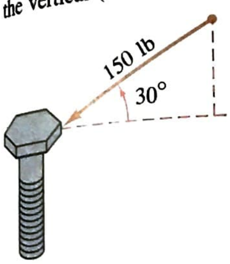

FIGURE FOR 57

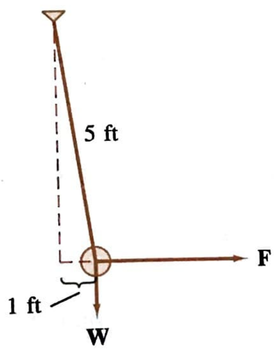

FIGURE FOR 58

59. To carry a 100-pound cylindrical weight, two men lift on the ends of short ropes tied to an eyelet on the top center of the cylinder. If one rope makes a 20° angle away from the vertical and the other a 30° angle, find the following.
(a) the tension in each rope if the resultant force is vertical
(b) the vertical component of each man's force (see figure)

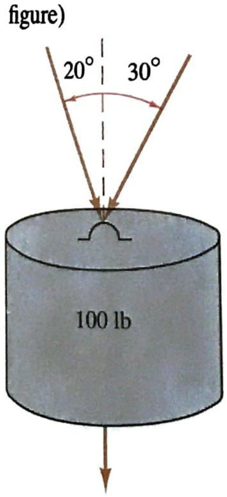

FIGURE FOR 59

60. An airplane is headed 32° north of west. Its speed with respect to the air is 580 miles per hour. The wind at the plane's altitude is from the southwest at 60 miles per hour. What is the true direction of the plane, and what is its speed with respect to the ground? (See figure.)

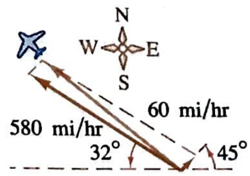

FIGURE FOR 60

61. A plane flies at a constant ground speed of 450 miles per hour due east and encounters a 50 mile per hour wind from the northwest. Find the air speed and compass direction that will allow the plane to maintain its ground speed and eastward direction.

62. A ball is thrown into the air with an initial velocity of 80 feet per second and at an angle of 50° with the horizontal. Find the vertical and horizontal components of the initial velocity.

63. Three vertices of a parallelogram are (1, 2), (3, 1), and (8, 4). Find the three possible fourth vertices (see figure).

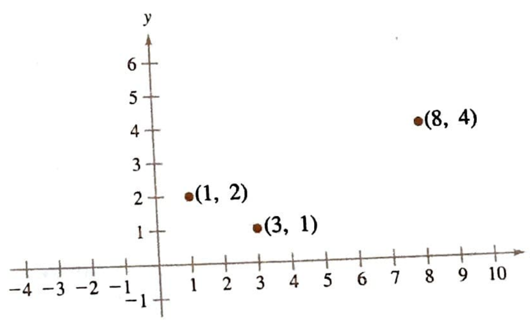

FIGURE FOR 63

64. Use vectors to find the points of trisection of the line segment with endpoints (1, 2) and (7, 5).

65. Prove that the vectors

$$\mathbf{u} = (\cos\theta)\mathbf{i} - (\sin\theta)\mathbf{j}$$

and

$$\mathbf{v} = (\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}$$

are unit vectors for any angle θ.

66. Using vectors, prove that the line segment joining the midpoints of two sides of a triangle is parallel to and one-half the length of the third side.

67. Using vectors, prove that the diagonals of a parallelogram bisect each other.

EXERCISES for Section 13.2

In Exercises 1–4, find (a) $\mathbf{u} \cdot \mathbf{v}$, (b) $\mathbf{u} \cdot \mathbf{u}$, (c) $\|\mathbf{u}\|^2$, (d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v}$, and (e) $\mathbf{u} \cdot 2\mathbf{v}$.

1. $\mathbf{u} = \langle 3, 4\rangle$, $\mathbf{v} = \langle 2, -3\rangle$
2. $\mathbf{u} = \langle 5, 12\rangle$, $\mathbf{v} = \langle -3, 2\rangle$
3. $\mathbf{u} = \mathbf{i} - 4\mathbf{j}$, $\mathbf{v} = \frac{1}{2}\mathbf{i} + 3\mathbf{j}$
4. $\mathbf{u} = \mathbf{i}$, $\mathbf{v} = \mathbf{i}$

In Exercises 5–12, find the angle θ between the given vectors.

5. $\mathbf{u} = \langle 1, 1\rangle$, $\mathbf{v} = \langle 2, -2\rangle$
6. $\mathbf{u} = \langle 3, 1\rangle$, $\mathbf{v} = \langle 2, -1\rangle$
7. $\mathbf{u} = \langle 1, 1\rangle$, $\mathbf{v} = \langle 3, -1\rangle$
8. $\mathbf{u} = \langle 1, 2\rangle$, $\mathbf{v} = \langle 2, -1\rangle$
9. $\mathbf{u} = \langle 2, -3\rangle$, $\mathbf{v} = \langle -9, -6\rangle$
10. $\mathbf{u} = \langle -1, 2\rangle$, $\mathbf{v} = \langle 4, 6\rangle$
11. $\mathbf{u} = \cos\frac{\pi}{6}\mathbf{i} + \sin\frac{\pi}{6}\mathbf{j}$

    $\mathbf{v} = \cos\frac{3\pi}{4}\mathbf{i} + \sin\frac{3\pi}{4}\mathbf{j}$
12. $\mathbf{u} = \cos\frac{2\pi}{3}\mathbf{i} + \sin\frac{2\pi}{3}\mathbf{j}$

    $\mathbf{v} = \cos\frac{\pi}{12}\mathbf{i} + \sin\frac{\pi}{12}\mathbf{j}$

In Exercises 13–16, find the positive angle θ between the given vector and the positive x-axis.

13. $\mathbf{u} = \mathbf{i} - \mathbf{j}$
14. $\mathbf{u} = \sqrt{3}\mathbf{i} + \mathbf{j}$
15. $\mathbf{u} = \langle 3, 2\rangle$
16. $\mathbf{u} = \langle 12, 5\rangle$

In Exercises 17–24, determine whether $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, parallel, or neither.

17. $\mathbf{u} = \langle 4, 0\rangle$, $\mathbf{v} = \langle 1, 1\rangle$
18. $\mathbf{u} = \langle 2, -4\rangle$, $\mathbf{v} = \langle 2, 1\rangle$
19. $\mathbf{u} = \langle 2, 18\rangle$, $\mathbf{v} = \left\langle \frac{3}{2}, -\frac{1}{6}\right\rangle$
20. $\mathbf{u} = \langle 0, 4\rangle$, $\mathbf{v} = \langle \sqrt{3}, 1\rangle$
21. $\mathbf{u} = \langle 6, -4\rangle$, $\mathbf{v} = \langle -3, 2\rangle$
22. $\mathbf{u} = \left\langle -\frac{1}{3}, \frac{2}{3}\right\rangle$, $\mathbf{v} = \langle 2, -4\rangle$
23. $\mathbf{u} = \mathbf{i} - 3\mathbf{j}$, $\mathbf{v} = 3\mathbf{i} - \mathbf{j}$
24. $\mathbf{u} = 4\mathbf{i} + 3\mathbf{j}$, $\mathbf{v} = \frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j}$

In Exercises 25–30, (a) find the projection of $\mathbf{u}$ onto $\mathbf{v}$, and (b) find the vector component of $\mathbf{u}$ orthogonal to $\mathbf{v}$.

25. $\mathbf{u} = \langle 2, 3\rangle$, $\mathbf{v} = \langle 5, 1\rangle$
26. $\mathbf{u} = \langle 1, -2\rangle$, $\mathbf{v} = \langle 1, 3\rangle$
27. $\mathbf{u} = \langle 1, 1\rangle$, $\mathbf{v} = \langle 5, 0\rangle$
28. $\mathbf{u} = \langle 2, -3\rangle$, $\mathbf{v} = \langle 5, -1\rangle$
29. $\mathbf{u} = \langle 2, -3\rangle$, $\mathbf{v} = \langle 3, 2\rangle$
30. $\mathbf{u} = \langle \sqrt{3}, 1\rangle$, $\mathbf{v} = \langle -\sqrt{3}, -1\rangle$

31. A truck with a gross weight of 32,000 pounds is parked on a 15° slope (see figure). Assuming the only force to overcome is that due to gravity, find the following.
    (a) the force required to keep the truck from rolling down the hill
    (b) the force perpendicular to the hill
32. Rework Exercise 31 for a truck that is parked on a 16° slope.

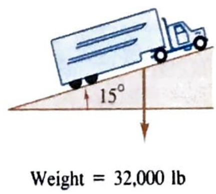

FIGURE FOR 31

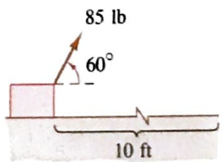

FIGURE FOR 33

33. An object is dragged 10 feet across a floor, using a force of 85 pounds. Find the work done if the direction of the force is 60° above the horizontal (see figure).

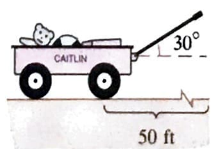

FIGURE FOR 34

34. A toy wagon is pulled by exerting a force of 15 pounds on a handle that makes a 30° angle with the horizontal. Find the work done in pulling the wagon 50 feet (see figure).
35. What is known about θ, the angle between two vectors $\mathbf{u}$ and $\mathbf{v}$, if
    (a) $\mathbf{u} \cdot \mathbf{v} = 0$ (b) $\mathbf{u} \cdot \mathbf{v} > 0$ (c) $\mathbf{u} \cdot \mathbf{v} < 0$?
36. What can be said about the vectors $\mathbf{u}$ and $\mathbf{v}$ if
    (a) the projection of $\mathbf{u}$ onto $\mathbf{v}$ equals $\mathbf{u}$?
    (b) the projection of $\mathbf{u}$ onto $\mathbf{v}$ equals $\mathbf{0}$?
37. Use vectors to prove that the diagonals of a rhombus are perpendicular.

As with real-valued functions, we can narrow the family of antiderivatives of a vector-valued function $\mathbf{r}'$ down to a single antiderivative by imposing an initial condition on the vector-valued function $\mathbf{r}$. This is demonstrated in the next example.

EXAMPLE 10 Finding the antiderivative of a vector-valued function

Find the antiderivative of

$$\mathbf{r}'(t) = \cos 2t\,\mathbf{i} - 2\sin t\,\mathbf{j}$$

that satisfies the initial condition $\mathbf{r}(0) = 3\mathbf{i} - 2\mathbf{j}$.

SOLUTION

$$\begin{aligned}\mathbf{r}(t) &= \int \mathbf{r}'(t)\,dt\\ &= \left(\int \cos 2t\,dt\right)\mathbf{i} - \left(2\int \sin t\,dt\right)\mathbf{j}\\ &= \left(\frac{1}{2}\sin 2t + C_1\right)\mathbf{i} + (2\cos t + C_2)\mathbf{j}\end{aligned}$$

Letting $t = 0$ and using the fact that $\mathbf{r}(0) = 3\mathbf{i} - 2\mathbf{j}$, we have

$$\mathbf{r}(0) = (0 + C_1)\mathbf{i} + (2 + C_2)\mathbf{j} = 3\mathbf{i} - 2\mathbf{j}.$$

Equating corresponding components produces $C_1 = 3$ and $C_2 = -4$. Thus, the antiderivative that satisfies the given initial condition is

$$\mathbf{r}(t) = \left(\frac{1}{2}\sin 2t + 3\right)\mathbf{i} + (2\cos t - 4)\mathbf{j}.$$

EXERCISES for Section 13.3

In Exercises 1–4, find the domain of the given vector-valued function.

1. $\mathbf{r}(t) = 5t\,\mathbf{i} - \dfrac{1}{t}\mathbf{j}$
2. $\mathbf{r}(t) = \sqrt{4 - t^2}\,\mathbf{i} + t^2\,\mathbf{j}$
3. $\mathbf{r}(t) = e^t\mathbf{i} + \ln t\,\mathbf{j}$
4. $\mathbf{r}(t) = \dfrac{1}{t-3}\mathbf{i} + \dfrac{1}{t-5}\mathbf{j}$

In Exercises 5 and 6, find $\|\mathbf{r}(t)\|$.

5. $\mathbf{r}(t) = \sin \pi t\,\mathbf{i} + \cos \pi t\,\mathbf{j}$
6. $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + 3t\,\mathbf{j}$

In Exercises 7–10, sketch the curve represented by the vector-valued function and give the orientation of the curve.

7. $\mathbf{r}(t) = 3t\,\mathbf{i} + (t - 1)\mathbf{j}$
8. $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$
9. $\mathbf{r}(t) = t\mathbf{i} + t^2\,\mathbf{j}$
10. $\mathbf{r}(t) = t\mathbf{i} + \dfrac{1}{t}\mathbf{j}$

In Exercises 11–16, evaluate the limit.

11. $\displaystyle\lim_{t\to 3}\left(t\mathbf{i} + \frac{t^2 - 9}{t^2 - 3t}\mathbf{j}\right)$
12. $\displaystyle\lim_{t\to 0}\left(e^t\mathbf{i} + \frac{\sin t}{t}\mathbf{j}\right)$

13. $\lim_{t \to 0} \left(\frac{1 - \cos t}{t} \mathbf{i} + t^2 \mathbf{j} \right)$

14. $\lim_{t \to 1} \left(\sqrt{t}\,\mathbf{i} + \frac{\ln t}{t^2 - 1} \mathbf{j} \right)$

15. $\lim_{t \to \infty} \left(\frac{2}{t^2} \mathbf{i} + e^{-2t} \mathbf{j} \right)$

16. $\lim_{t \to \infty} \left(\frac{3t^2}{t^2 + 1} \mathbf{i} + \frac{5}{t} \mathbf{j} \right)$

In Exercises 17–20, determine the interval(s) on which the vector-valued function is continuous.

17. $\mathbf{r}(t) = t\,\mathbf{i} + \frac{1}{t}\mathbf{j}$
18. $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + \sqrt{t - 1}\,\mathbf{j}$
19. $\mathbf{r}(t) = \ln t\,\mathbf{i} + e^t\,\mathbf{j}$
20. $\mathbf{r}(t) = \arccos t\,\mathbf{i} + t^2\,\mathbf{j}$

In Exercises 21–24, (a) sketch the plane curve represented by the vector-valued function, and (b) sketch the vectors $\mathbf{r}(t_0)$ and $\mathbf{r}'(t_0)$ for the specified value of t_0. Position the vectors so that the initial point of $\mathbf{r}(t_0)$ is at the origin and the initial point of $\mathbf{r}'(t_0)$ is at the terminal point of $\mathbf{r}(t_0)$.

21. $\mathbf{r}(t) = t^2\,\mathbf{i} + t\,\mathbf{j}$ $\quad t_0 = 2$
22. $\mathbf{r}(t) = t\,\mathbf{i} + t^3\,\mathbf{j}$ $\quad t_0 = 1$
23. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$ $\quad t_0 = \frac{\pi}{2}$
24. $\mathbf{r}(t) = t^2\,\mathbf{i} + \frac{1}{t}\mathbf{j}$ $\quad t_0 = 2$

In Exercises 25–30, find $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$.

25. $\mathbf{r}(t) = 3t\,\mathbf{i} + (t - 1)\,\mathbf{j}$
26. $\mathbf{r}(t) = \frac{1}{t}\mathbf{i} + \frac{t + 1}{t - 1}\mathbf{j}$
27. $\mathbf{r}(t) = \langle a \cos t, a \sin t \rangle$
28. $\mathbf{r}(t) = \langle \sec t, \tan t \rangle$
29. $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t \rangle$
30. $\mathbf{r}(t) = \langle \cot t, 2 \sin t \cos t \rangle$

In Exercises 31 and 32, find the following.
(a) $\mathbf{r}'(t)$
(b) $D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)]$
(c) $D_t[3\mathbf{r}(t) - \mathbf{u}(t)]$
(d) $D_t[\|\mathbf{r}(t)\|]$

31. $\mathbf{r}(t) = 3t\,\mathbf{i} + 4t\,\mathbf{j}$, $\mathbf{u}(t) = 4t\,\mathbf{i} + t^2\,\mathbf{j}$
32. $\mathbf{r}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j}$, $\mathbf{u}(t) = \cos t\,\mathbf{i} - \sin t\,\mathbf{j}$

In Exercises 33–38, find the open interval(s) on which the curve given by the vector-valued function is smooth.

33. $\mathbf{r}(t) = t^2\,\mathbf{i} + t^3\,\mathbf{j}$
34. $\mathbf{r}(t) = \frac{1}{t - 1}\mathbf{i} + 3t\,\mathbf{j}$
35. $\mathbf{r}(\theta) = \langle 2 \cos^3 \theta, 3 \sin^3 \theta \rangle$
36. $\mathbf{r}(\theta) = \langle \theta + \sin \theta, 1 - \cos \theta \rangle$
37. $\mathbf{r}(\theta) = \langle \theta - 2 \sin \theta, 1 - 2 \cos \theta \rangle$
38. $\mathbf{r}(t) = \left\langle \frac{3t}{1 + t^3}, \frac{3t^2}{1 + t^3} \right\rangle$

In Exercises 39–42, evaluate the indefinite integral.

39. $\int (6t^2\,\mathbf{i} + 3\,\mathbf{j})\,dt$
40. $\int \left(\frac{1}{t}\mathbf{i} + e^t\,\mathbf{j} \right) dt$
41. $\int (4 \sin t\,\mathbf{i} + 3 \cos t\,\mathbf{j})\,dt$
42. $\int (te^{-t^2}\mathbf{i} + t\,\mathbf{j})\,dt$

In Exercises 43–46, evaluate the definite integral.

43. $\int_0^1 (6t\,\mathbf{i} - 3t\,\mathbf{j})\,dt$
44. $\int_0^1 (\sqrt{t}\,\mathbf{i} + \sqrt{t + 1}\,\mathbf{j})\,dt$
45. $\int_0^{\pi/2} (3 \cos t\,\mathbf{i} + 3 \sin t\,\mathbf{j})\,dt$
46. $\int_0^3 (e^t\,\mathbf{i} + te^t\,\mathbf{j})\,dt$

In Exercises 47–50, find $\mathbf{r}(t)$ for the given conditions.

47. $\mathbf{r}'(t) = 4e^{2t}\,\mathbf{i} + 3e^t\,\mathbf{j}$
$\mathbf{r}(0) = 2\mathbf{i}$
48. $\mathbf{r}'(t) = 2t\,\mathbf{i} + \sqrt{t}\,\mathbf{j}$
$\mathbf{r}(0) = \mathbf{i} + \mathbf{j}$
49. $\mathbf{r}''(t) = -32\mathbf{j}$
$\mathbf{r}'(0) = 600\sqrt{3}\,\mathbf{i} + 600\mathbf{j}$, $\mathbf{r}(0) = \mathbf{0}$
50. $\mathbf{r}''(t) = -4 \cos t\,\mathbf{i} - 3 \sin t\,\mathbf{j}$
$\mathbf{r}'(0) = 3\mathbf{j}$, $\mathbf{r}(0) = 4\mathbf{i}$

In Exercises 51–55, prove the given property. In each case assume that $\mathbf{r}$ and $\mathbf{u}$ are differentiable vector-valued functions of t, f is a differentiable real-valued function of t, and c is a scalar.

51. $D_t[c\mathbf{r}(t)] = c\mathbf{r}'(t)$
52. $D_t[\mathbf{r}(t) \pm \mathbf{u}(t)] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$
53. $D_t[f(t)\mathbf{r}(t)] = f(t)\mathbf{r}'(t) + f'(t)\mathbf{r}(t)$
54. $D_t[\mathbf{r}(f(t))] = \mathbf{r}'(f(t))f'(t)$
55. If $\mathbf{r}(t) \cdot \mathbf{r}(t) = c$, then $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$.