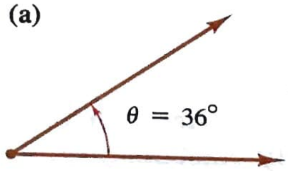
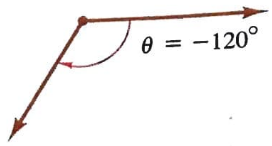
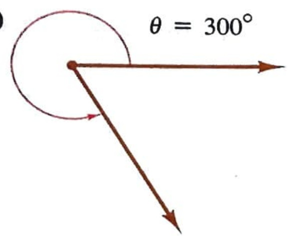
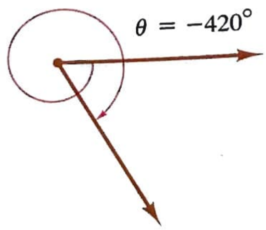
EXERCISES for Section 8.1

In Exercises 1 and 2, determine two coterminal angles (one positive and one negative) for the given angle. Give your answers in degrees.

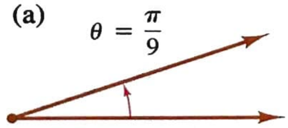
1. (a)


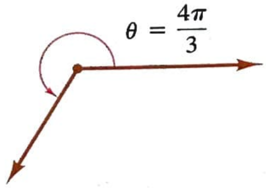
(b)


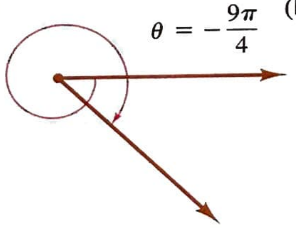
2. (a)


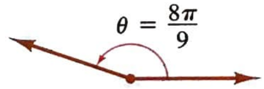
(b)


In Exercises 3 and 4, determine two coterminal angles (one positive and one negative) for the given angle. Give your answers in radians.

3. (a) $\theta = \dfrac{\pi}{9}$


(b) $\theta = \dfrac{4\pi}{3}$


4. (a) $\theta = -\dfrac{9\pi}{4}$


(b) $\theta = \dfrac{8\pi}{9}$


In Exercises 5 and 6, express the given angle in radian measure as a multiple of π.

5. (a) 30° (b) 150°
(c) 315° (d) 120°

6. (a) −20° (b) −240°
(c) −270° (d) 144°

In Exercises 7 and 8, express the given angle in degree measure.

7. (a) $\dfrac{3\pi}{2}$ (b) $\dfrac{7\pi}{6}$
(c) $-\dfrac{7\pi}{12}$ (d) $\dfrac{\pi}{9}$

8. (a) $\dfrac{7\pi}{3}$ (b) $-\dfrac{11\pi}{30}$
(c) $\dfrac{11\pi}{6}$ (d) $\dfrac{34\pi}{15}$

9. Let r represent the radius of a circle, θ the central angle (measured in radians), and s the length of the arc subtended by the angle. Use the relationship $\theta = s/r$ to complete the following table.

r	8 ft	15 in.	85 cm		
s	12 ft			96 in.	8642 mi
θ		1.6	$\dfrac{3\pi}{4}$	4	$\dfrac{2\pi}{3}$

10. The pointer on a voltmeter is 2 inches in length (see figure). Find the angle through which the pointer rotates when it moves $\frac{1}{2}$ inch on the scale.

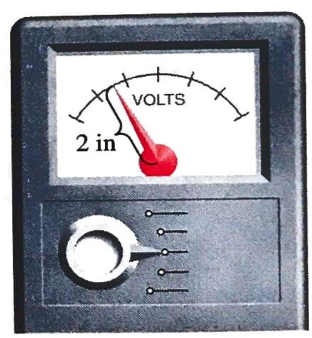


FIGURE FOR 10

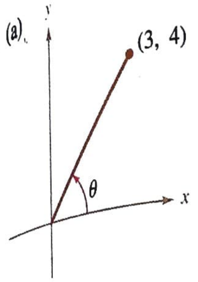
FIGURE FOR 11

11. An electric hoist is being used to lift a piece of equipment (see figure). The diameter of the drum on the hoist is 8 inches and the equipment must be raised one foot. Find the number of degrees through which the drum must rotate.

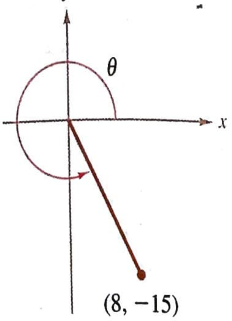
12. A car is moving at the rate of 50 miles per hour, and the diameter of its wheels is 2.5 feet.
(a) Find the number of revolutions per minute that the wheels are rotating.
(b) Find the angular speed of the wheels in radians per minute.

In Exercises 13 and 14, determine all six trigonometric functions for the given angle θ.

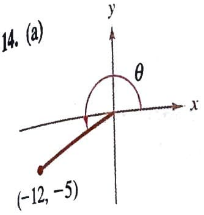
13. (a)



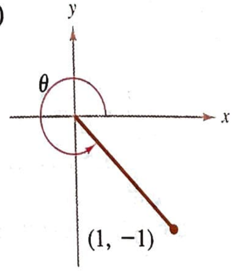
(b)



14. (a)



(b)



In Exercises 15 and 16, determine the quadrant in which θ lies.

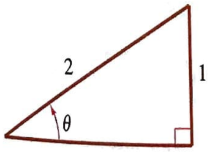
15. $\sin\theta < 0$ and $\cos\theta < 0$

16. $\sin\theta > 0$ and $\cos\theta < 0$

In Exercises 17–22, find the indicated trigonometric functions from the given one. (Assume $0 < \theta < \pi/2$.)

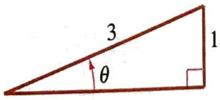
17. Given: $\sin\theta = \frac{1}{2}$

Find: $\csc\theta$



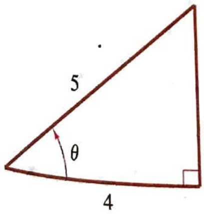
18. Given: $\sin\theta = \frac{1}{3}$

Find: $\tan\theta$



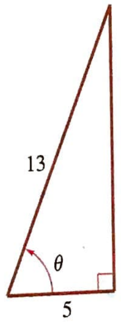
19. Given: $\cos\theta = \frac{4}{5}$

Find: $\cot\theta$



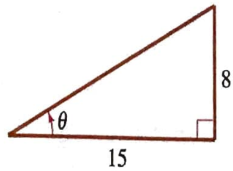
20. Given: $\sec\theta = \frac{13}{5}$

Find: $\cot\theta$



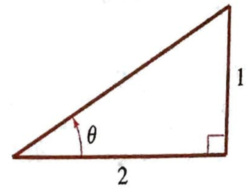
21. Given: $\cot\theta = \frac{15}{8}$

Find: $\sec\theta$



22. Given: $\tan\theta = \frac{1}{2}$

Find: $\sin\theta$



In Exercises 23–26, evaluate the sine, cosine, and tangent of the given angles *without* using a calculator.

23. (a) $60°$ (b) $\frac{2\pi}{3}$
(c) $\frac{\pi}{4}$ (d) $\frac{5\pi}{4}$

24. (a) $-\frac{\pi}{6}$ (b) $150°$
(c) $-\frac{\pi}{2}$ (d) $\frac{\pi}{2}$

25. (a) $225°$ (b) $-225°$
(c) $300°$ (d) $330°$

26. (a) $750°$ (b) $510°$
(c) $\frac{10\pi}{3}$ (d) $\frac{17\pi}{3}$

In Exercises 27–30, use a calculator to evaluate the given trigonometric functions to four significant digits.

27. (a) $\sin 10°$ (b) $\csc 10°$

28. (a) $\sec 225°$ (b) $\sec 135°$

29. (a) $\tan\frac{\pi}{9}$ (b) $\tan\frac{10\pi}{9}$

30. (a) $\cot(1.35)$ (b) $\tan(1.35)$

In Exercises 31–34, find two values of θ corresponding to the given functions. List the measure of θ in radians $(0 \le \theta < 2\pi)$. Do not use a calculator.

31. (a) $\cos\theta = \frac{\sqrt{2}}{2}$ (b) $\cos\theta = -\frac{\sqrt{2}}{2}$

32. (a) $\sec\theta = 2$ (b) $\sec\theta = -2$

33. (a) $\tan\theta = 1$ (b) $\cot\theta = -\sqrt{3}$

34. (a) $\sin\theta = \frac{\sqrt{3}}{2}$ (b) $\sin\theta = -\frac{\sqrt{3}}{2}$

In Exercises 35–42, solve the given equation for θ $(0 \le \theta < 2\pi)$. For some of the equations, you should use the trigonometric identities listed in this section.

35. $2\sin^2\theta = 1$ **36.** $\tan^2\theta = 3$

37. $\tan^2\theta - \tan\theta = 0$ **38.** $2\cos^2\theta - \cos\theta = 1$

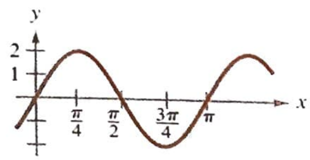
39. $\sec\theta\csc\theta = 2\csc\theta$ **40.** $\sin\theta = \cos\theta$

41. $\cos^2\theta + \sin\theta = 1$ **42.** $\cos(\theta/2) - \cos\theta = 1$

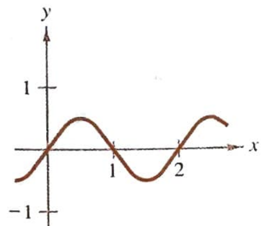
EXERCISES for Section 8.2

In Exercises 1–10, determine the period and amplitude of the given function.

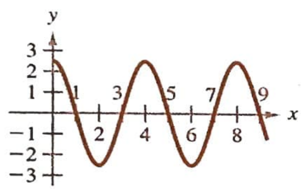
1. $y = 2 \sin 2x$



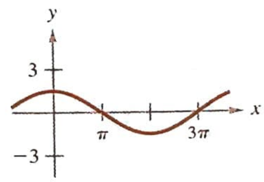
2. $y = \dfrac{1}{2} \sin \pi x$



3. $y = \dfrac{5}{2} \cos \dfrac{\pi x}{2}$



4. $y = \dfrac{3}{2} \cos \dfrac{x}{2}$



5. $y = -2 \sin \dfrac{x}{3}$

6. $y = -\cos \dfrac{2x}{3}$

7. $y = -2 \sin 10x$

8. $y = \dfrac{1}{2} \cos \dfrac{2x}{3}$

9. $y = 3 \sin 4\pi x$

10. $y = \dfrac{2}{3} \cos \dfrac{\pi x}{10}$

In Exercises 11 and 12, use a graphing utility to graph each function f on the same set of coordinate axes when $c = -2$, $c = -1$, $c = 1$, and $c = 2$. Write a description of the change in the graph caused by changing c.

11. (a) $f(x) = c \sin x$
(b) $f(x) = \cos(cx)$
(c) $f(x) = \cos(\pi x - c)$

12. (a) $f(x) = \sin x + c$
(b) $f(x) = -\sin(2\pi x - c)$
(c) $f(x) = c \cos x$

In Exercises 13–26, sketch the graph of the given function.

13. $y = \sin \dfrac{x}{2}$

14. $y = 2 \cos 2x$

15. $y = -2 \sin 6x$

16. $y = \cos 2\pi x$

17. $y = -\sin \dfrac{2\pi x}{3}$

18. $y = 2 \tan x$

19. $y = \csc \dfrac{x}{2}$

20. $y = \tan 2x$

21. $y = 2 \sec 2x$

22. $y = \csc 2\pi x$

23. $y = \sin (x + \pi)$

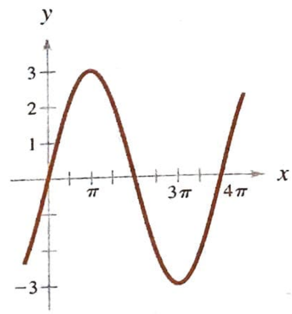
24. $y = \cos \left(2x - \dfrac{\pi}{3}\right)$

25. $y = 1 + \cos \left(x - \dfrac{\pi}{2}\right)$

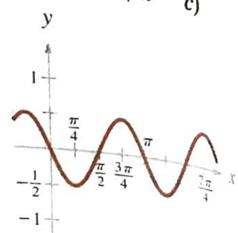
26. $y = -1 + \sin \left(x + \dfrac{\pi}{2}\right)$

In Exercises 27 and 28, find a, b, and c so that the graph of the function matches the graph in the figure.

27. $y = a \cos(bx - c)$



28. $y = a \sin(bx - c)$



In Exercises 29 and 30, use a graphing utility to compare the graph of f with the given graph. In each case, the graph of f is a Fourier approximation of the given graph. Try to improve the approximation by adding a term to $f(x)$. Use a graphing utility to verify that your new approximation is better than the original. Can you find other terms to add to make the approximation even better? What is the pattern? (In Exercise 29, sine terms can be used to improve the approximation and in Exercise 30, cosine terms can be used.)

29. $f(x) = \dfrac{4}{\pi}\left(\sin \pi x + \dfrac{1}{3} \sin 3\pi x\right)$

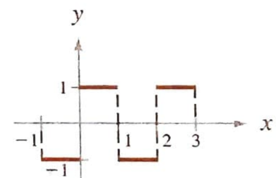


FIGURE FOR 29

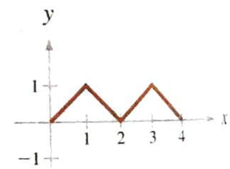


FIGURE FOR 30

30. $f(x) = \dfrac{1}{2} - \dfrac{4}{\pi^2}\left(\cos \pi x + \dfrac{1}{9} \cos 3\pi x\right)$

In Exercises 31–48, determine the limit (if it exists).

31. $\displaystyle\lim_{x \to 0} \frac{\sin x}{5x}$

32. $\displaystyle\lim_{x \to 0} \frac{3(1 - \cos x)}{x}$

33. $\displaystyle\lim_{\theta \to 0} \frac{\sec \theta - 1}{\theta \sec \theta}$

34. $\displaystyle\lim_{\theta \to 0} \frac{\cos \theta \tan \theta}{\theta}$

35. $\displaystyle\lim_{x \to 0} \frac{\sin^2 x}{x}$

36. $\displaystyle\lim_{\phi \to \pi} \phi \sec \phi$

37. $\displaystyle\lim_{x \to \pi/2} \frac{\cos x}{\cot x}$

38. $\displaystyle\lim_{x \to \pi/4} \frac{1 - \tan x}{\sin x - \cos x}$

39. $\displaystyle\lim_{t \to 0} \frac{\sin^2 t}{t^2}$ $\left[\text{Hint: Find } \displaystyle\lim_{t \to 0} \left(\frac{\sin t}{t}\right)^2.\right]$

40. $\displaystyle\lim_{t \to 0} \frac{\sin 3t}{t}$ $\left[\text{Hint: Find } \displaystyle\lim_{t \to 0} 3\left(\frac{\sin 3t}{3t}\right).\right]$

41. $\lim_{x\to 0} \frac{\sin 2x}{\sin 3x}$

$\left[\text{Hint: Find } \lim_{x\to 0} \left(\frac{2 \sin 2x}{2x}\right)\left(\frac{3x}{3 \sin 3x}\right)\cdot\right]$

42. $\lim_{x\to 0} \frac{\tan^2 x}{x}$

43. $\lim_{h\to 0} \frac{(1 - \cos h)^2}{h}$

44. $\lim_{h\to 0} (1 + \cos 2h)$

45. $\lim_{x\to \pi} \cot x$

46. $\lim_{x\to \pi/2} \sec x$

47. $\lim_{x\to 0^+} \frac{2}{\sin x}$

48. $\lim_{x\to \pi/2^+} \frac{-2}{\cos x}$

In Exercises 49–54, find the discontinuities (if any) for the function. Which are removable?

49. $f(x) = x + \sin x$

50. $f(x) = \cos \frac{\pi x}{2}$

51. $f(x) = \csc 2x$

52. $f(x) = \tan \frac{\pi x}{2}$

53. $f(x) = \begin{cases} \csc \frac{\pi x}{6}, & |x - 3| < 2 \\ 2, & |x - 3| \geq 2 \end{cases}$

54. $f(x) = \begin{cases} \tan \frac{\pi x}{4}, & |x| < 1 \\ x, & |x| \geq 1 \end{cases}$

In Exercises 55 and 56, use a computer or graphics calculator to sketch the graph of the function f, and find the specified limit (if it exists).

55. $f(x) = \frac{\sin 5x}{\sin 2x}$, $\quad \lim_{x\to 0} f(x)$

56. $f(x) = \frac{1 - \cos 3x}{2x}$, $\quad \lim_{x\to 0} f(x)$

In Exercises 57–62, use a graphing utility to graph the given function and the equations $y = x$ and $y = -x$ in the same viewing rectangle. Using the graphs to observe the Squeeze Theorem visually, find $\lim_{x\to 0} f(x)$.

57. $f(x) = x \cos x$

58. $f(x) = |x \sin x|$

59. $f(x) = |x| \sin x$

60. $f(x) = |x| \cos x$

61. $f(x) = x \sin \frac{1}{x}$

62. $h(x) = x \cos \frac{1}{x}$

63. Use a graphing utility to graph $f(x) = x$, $g(x) = \sin x$, and $h(x) = \sin x/x$ in the same viewing rectangle. Compare the magnitudes of $f(x)$ and $g(x)$ when x is "close to" 0. Use the comparison to write a short paragraph explaining why $\lim_{x\to 0} h(x) = 1$.

64. Use a graphing utility to graph $f(x) = x$, $g(x) = \sin^2 x$, and $h(x) = \sin^2 x/x$ in the same viewing rectangle. Compare the magnitudes of $f(x)$ and $g(x)$ when x is "close to" 0. Use the comparison to write a short paragraph explaining why $\lim_{x\to 0} h(x) = 0$.

65. Find the limit as x approaches zero of the ratio of the triangle's area to the total shaded area in the figure.

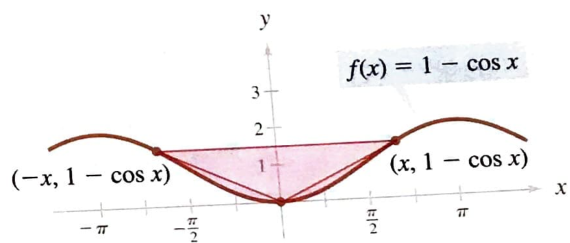


FIGURE FOR 65

66. When tuning a piano, a technician strikes a tuning fork for the A above middle C and sets up a wave motion that can be approximated by

$$y = 0.001 \sin 880\pi t$$

where t is the time in seconds.
(a) What is the period p of this function?
(b) What is the frequency f of this note? ($f = 1/p$)
(c) Sketch the graph of this function.

67. Prove the second part of Theorem 8.2.

8.3 Derivatives of Trigonometric Functions

Derivatives of sine and cosine functions ▪ Derivatives of other trigonometric functions ▪ Applications

In the previous section, we discussed the following limits.

$$\lim_{\Delta x\to 0} \frac{\sin \Delta x}{\Delta x} = 1 \quad \text{and} \quad \lim_{\Delta x\to 0} \frac{1 - \cos \Delta x}{\Delta x} = 0$$

These two limits are crucial in the proofs of the derivatives of the sine and cosine functions. The derivatives of the other four trigonometric func-
these two.

Choosing the positive solution, we have $x = 8$. Finally, when $\theta = \pi/3$ and $x = 8$, the velocity of the piston is

$$\frac{dx}{dt} = \frac{6(8)(\sqrt{3}/2)}{6(1/2) - 16}(400\pi) = \frac{9600\pi\sqrt{3}}{-13} \approx -4018 \text{ in./min.}$$

REMARK Note that the velocity in Example 12 is negative because the piston is moving to the left.

EXERCISES for Section 8.3

In Exercises 1–44, find the derivative of the given function.

1. $y = x^2 - \frac{1}{2}\cos x$

2. $y = 5 + \sin x$

3. $y = \frac{1}{x} - 3\sin x$

4. $g(t) = \pi \cos t$

5. $f(x) = 4\sqrt{x} + 3\cos x$

6. $f(x) = 2\sin x + 3\cos x$

7. $f(t) = t^2 \sin t$

8. $f(x) = \frac{\sin x}{x}$

9. $g(t) = \frac{\cos t}{t}$

10. $f(\theta) = (\theta + 1)\cos \theta$

11. $y = \tan x - x$

12. $y = x + \cot x$

13. $y = 5x \csc x$

14. $y = \frac{\sec x}{x}$

15. $f(\theta) = -\csc \theta - \sin \theta$

16. $h(x) = x \sin x + \cos x$

17. $g(t) = t^2 \sin t + 2t \cos t - 2 \sin t$

18. $h(\theta) = 5 \sec 3\theta + \tan 3\theta$

19. $f(x) = \sin \pi x \cos \pi x$

20. $f(x) = \tan 2x \cot 2x$

21. $y = \frac{1 + \csc x}{1 - \csc x}$

22. $y = \frac{\sin \theta}{1 - \cos \theta}$

23. $y = \cos 3x$

24. $y = \sin 2x$

25. $y = 3 \tan 4x$

26. $y = 2 \cos \frac{x}{2}$

27. $y = \sin \pi x$

28. $y = \sec x^2$

29. $y = \frac{1}{4}\sin^2 x$

30. $y = 5\cos^2 \pi x$

31. $y = \frac{1}{4}\sin^2 2x$

32. $y = 5\cos(\pi x)^2$

33. $y = \sqrt{\sin x}$

34. $y = \csc^2 4x$

35. $y = \sec^3 2x$

36. $y = x^2 \sin \frac{1}{x}$

37. $y = \ln|\csc x - \cot x|$

38. $y = \ln|\sec x + \tan x|$

39. $y = e^x(\sin x + \cos x)$

40. $y = \tan^2 e^x$

41. $y = e^{\tan x}$

42. $y = \ln|\cot x|$

43. $y = \ln|\tan x|$

44. $y = \ln|\sin x|$

In Exercises 45–50, use implicit differentiation to find dy/dx and evaluate the derivative at the indicated point.

Equation	*Point*
45. $\sin x + \cos 2y = 2$	$\left(\frac{\pi}{2}, 0\right)$
46. $2 \sin x \cos y = 1$	$\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$
47. $\tan(x + y) = x$	$(0, 0)$
48. $\cot y = x - y$	$\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$
49. $x \cos y = 1$	$\left(2, \frac{\pi}{3}\right)$
50. $x = \sec \frac{1}{y}$	$\left(\sqrt{2}, \frac{\pi}{4}\right)$

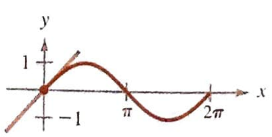
In Exercises 51 and 52, show that the function satisfies the differential equation.

51. $y = 2\sin x + 3\cos x$
$y'' + y = 0$

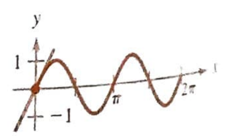
52. $y = e^x(\cos \sqrt{2}x + \sin \sqrt{2}x)$
$y'' - 2y' + 3y = 0$

In Exercises 53 and 54, find the slope of the tangent line to the given sine function at the origin. Compare this value to the number of complete cycles in the interval $[0, 2\pi]$.

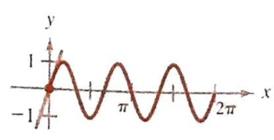
53. (a) $y = \sin x$



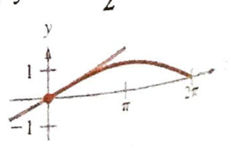
(b) $y = \sin 2x$



54. (a) $y = \sin 3x$



(b) $y = \sin \frac{x}{2}$



In Exercises 55–62, evaluate each limit, using L'Hôpital's Rule when necessary.

55. $\lim_{x\to 0} \dfrac{\sin 2x}{\sin 3x}$

56. $\lim_{x\to \pi} \dfrac{\sin x}{x-\pi}$

57. $\lim_{x\to 0} \dfrac{x-\tan x}{x-\sin x}$

58. $\lim_{\theta\to 0} \dfrac{1-\cos\theta}{\theta}$

59. $\lim_{\theta\to 0} \dfrac{1-\cos 2\theta}{4\theta^2}$

60. $\lim_{x\to \pi/4} (\tan 2x - \sec 2x)$

61. $\lim_{x\to \infty} x \sin \dfrac{1}{x}$

62. $\lim_{x\to 0} \dfrac{1-e^x}{\sin x}$

In Exercises 63–66, sketch the graph of each function on the indicated interval, making use of relative extrema and points of inflection.

	Function	*Interval*
63.	$f(x) = 2\sin x + \sin 2x$	$[0, 2\pi]$
64.	$f(x) = 2\sin x + \cos 2x$	$[0, 2\pi]$
65.	$f(x) = x - \sin x$	$[0, 4\pi]$
66.	$f(x) = \cos x - x$	$[0, 4\pi]$

In Exercises 67 and 68, sketch the graph of the function on the interval $[-\pi, \pi]$. In each case use Newton's Method to approximate the critical number to two decimal places.

67. $f(x) = x\sin x$

68. $f(x) = x\cos x$

69. The height of a weight oscillating on a spring is given by the equation $y = \frac{1}{3}\cos 12t - \frac{1}{4}\sin 12t$, where y is measured in inches and t is measured in seconds.
(a) Calculate the height and velocity of the weight when $t = \pi/8$ seconds.
(b) Show that the maximum displacement of the weight is $\frac{5}{12}$ inches.
(c) Find the period P of y. Find the frequency f (number of oscillations per second) if $f = 1/P$.

70. The general equation giving the height of an oscillating weight attached to a spring is

$$y = A\sin\left(\sqrt{\frac{k}{m}}t\right) + B\cos\left(\sqrt{\frac{k}{m}}t\right)$$

where k is the spring constant and m is the mass of the weight.
(a) Show that the maximum displacement of the weight is $\sqrt{A^2+B^2}$.
(b) Show that the frequency (number of oscillations per second) is $(1/2\pi)\sqrt{k/m}$. How is the frequency changed if the stiffness k of the spring is increased? How is the frequency changed if the mass m of the weight is increased?

71. A component is designed to slide a block of steel of weight W across a table and into a chute (see figure). The motion of the block is resisted by a frictional force proportional to its net weight. (Let k be the constant of proportionality.) Find the minimum force F needed to slide the block and find the corresponding value of θ. [*Hint:* $F\cos\theta$ is the force in the direction of motion and $F\sin\theta$ is the amount of force tending to lift the block. Therefore, the net weight of the block is $W - F\sin\theta$.]

72. When light waves traveling in a transparent medium strike the surface of a second transparent medium, they tend to "bend." This tendency is called refraction, and is given by **Snell's Law of Refraction,**

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$$

where θ_1 and θ_2 are the magnitudes of the angles shown in the accompanying figure, and v_1 and v_2 are the velocities of light in the two media. Show that light waves traveling from P to Q follow the path of minimum time.

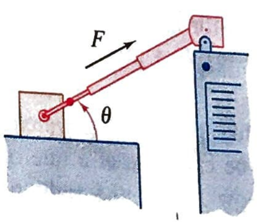


FIGURE FOR 71

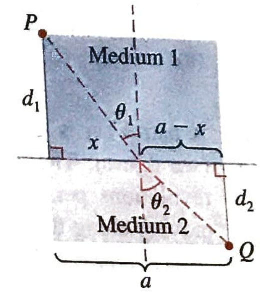


FIGURE FOR 72

73. A sector with central angle θ is cut from a circle of radius 12 inches, and the resulting edges are brought together to form a cone (see figure). Find the magnitude of θ so that the volume of the cone is maximum.

74. The cross sections of an irrigation canal are isosceles trapezoids, where the length of the three sides is 8 feet (see figure). Determine the angle of elevation θ of the sides so that the area of the cross section is maximum.

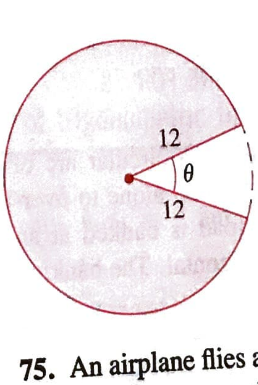


FIGURE FOR 73

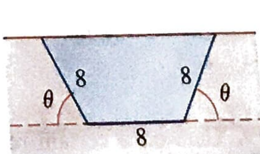


FIGURE FOR 74

75. An airplane flies at an altitude of 5 miles toward a point directly over an observer (see figure on next page). The speed of the plane is 600 miles per hour. Find the rate at which the angle of elevation θ is changing for the following angles.
(a) $\theta = 30°$ (b) $\theta = 60°$ (c) $\theta = 75°$.