EXERCISES for Section 7.1

In Exercises 1 and 2, evaluate each expression.

1. (a) $25^{3/2}$ (b) $81^{1/2}$
   (c) 3^{-2} (d) $27^{-1/3}$
2. (a) $64^{1/3}$ (b) 5^{-4}
   (c) $\left(\frac{1}{8}\right)^{1/3}$ (d) $\left(\frac{1}{4}\right)^{3}$

In Exercises 3–6, use the properties of exponents to simplify each expression.

3. (a) $(5^2)(5^3)$ (b) $(5^2)(5^{-3})$
   (c) $\frac{5^3}{25^2}$ (d) $\left(\frac{1}{4}\right)^2 2^6$
4. (a) $(4^2)^3$ (b) $(6^4)^{1/2}$
   (c) $[(8^{-1})(8^{2/3})]^3$ (d) $(32^{3/2})(4^2)$
5. (a) $e^2(e^4)$ (b) $(e^3)^4$
   (c) $(e^3)^{-2}$ (d) $\frac{e^5}{e^3}$
6. (a) $\left(\frac{1}{e}\right)^{-2}$ (b) $\left(\frac{e^5}{e^2}\right)^{-1}$
   (c) e^0 (d) $\frac{1}{e^{-3}}$

In Exercises 7–16, solve for x.

7. $3^x = 81$
8. $5^{x+1} = 125$
9. $\left(\frac{1}{3}\right)^{x-1} = 27$
10. $\left(\frac{1}{5}\right)^{2x} = 625$
11. $4^3 = (x+2)^3$
12. $18^2 = (5x-7)^2$
13. $x^{3/4} = 8$
14. $(x+3)^{4/3} = 16$
15. $e^{-2x} = e^5$
16. $e^x = 1$

In Exercises 17 and 18, compare the given number to the number e. Is the number less than or greater than e?

17. $\left(1 + \frac{1}{1{,}000{,}000}\right)^{1{,}000{,}000}$
18. $1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040}$

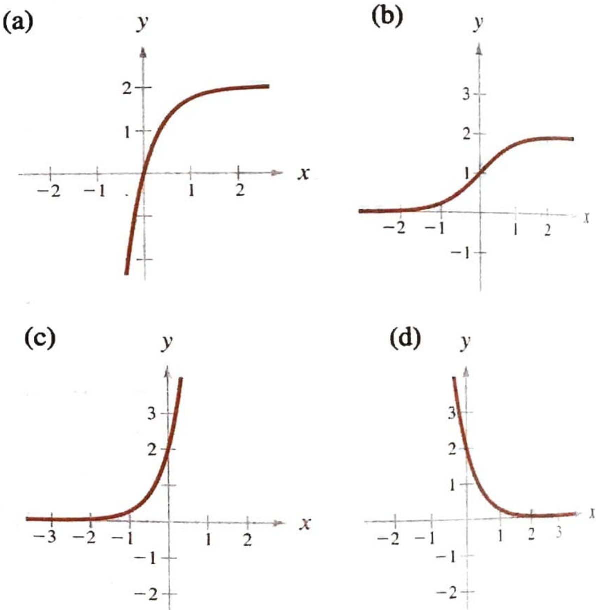
In Exercises 19–28, sketch the graph of the given function.

19. $y = 3^x$
20. $y = 3^{x-1}$
21. $y = \left(\frac{1}{3}\right)^x$
22. $y = 2^{x^2}$
23. $f(x) = 3^{-x^2}$
24. $f(x) = 3^{|x|}$
25. $h(x) = e^{x-2}$
26. $g(x) = -e^{x/2}$
27. $y = e^{-x^2}$
28. $y = e^{-x/2}$

29. Use a graphing utility to graph $f(x) = e^x$ and the given function on the same viewing rectangle. How are the two graphs related?
    (a) $g(x) = f(x-2) = e^{x-2}$
    (b) $h(x) = -\frac{1}{2}f(x) = -\frac{1}{2}e^x$
    (c) $q(x) = f(-x) + 3 = e^{-x} + 3$
30. Use a graphing utility to graph the function. Use the graph to determine any asymptotes of the function.
    (a) $f(x) = \frac{8}{1 + e^{-0.5x}}$ (b) $g(x) = \frac{8}{1 + e^{-0.5/x}}$

In Exercises 31–34, match the equation with the correct graph. Assume that a and C are arbitrary real numbers such that $a > 0$. [The graphs are labeled (a)–(d).]

31. $y = Ce^{ax}$
32. $y = Ce^{-ax}$
33. $y = C(1 - e^{-ax})$
34. $y = \frac{C}{1 + e^{-ax}}$



In Exercises 35 and 36, find the amount of an investment of P dollars invested at r percent for t years if the interest is compounded (a) annually, (b) semiannually, (c) monthly, (d) daily, and (e) continuously.

35. $P = \$1000$, $r = 10\%$, $t = 10$ years
36. $P = \$2500$, $r = 12\%$, $t = 20$ years

In Exercises 37 and 38, find the investment that would be required at r percent compounded continuously to yield an amount of $100,000 in (a) 1 year, (b) 10 years, (c) 20 years, and (d) 50 years.

37. $r = 12\%$
38. $r = 9\%$

39. The average time between incoming calls at a switchboard is 3 minutes. If a call has just come in, the probability that the next call will come within the next t minutes is given by

$$P(t) = 1 - e^{-t/3}.$$

Find (a) $P(\frac{1}{2})$, (b) $P(2)$, and (c) $P(5)$.

40. A certain automobile gets 28 mi/gal at speeds of up to 50 mi/hr. At speeds of over 50 mi/hr, the number of miles per gallon drops at the rate of 12% for each 10 mi/hr. If s is the speed (in miles per hour) and y is the miles per gallon, then

$$y = 28e^{0.6-0.012s}, \quad s \geq 50.$$

Use this function to complete the following table.

Speed (s)	50	55	60	65	70
Miles per gallon (y)					

41. The population of a bacterial culture is given by the logistics growth function

$$y = \frac{850}{1 + e^{-0.2t}}$$

where y is the number of bacteria and t is the time in days.
(a) Find the limit of this function as t approaches infinity.
(b) Sketch the graph of this function.

42. The yield V (in millions of cubic feet per acre) for a forest at age t years is given by

$$V = 6.7e^{-48/t}.$$

(a) Find the volume per acre when $t = 20$ years and $t = 50$ years.
(b) Find the limiting volume of wood per acre as t approaches infinity.
(c) Sketch the graph of this function.

43. In a group project in learning theory, a mathematical model for the proportion P of correct responses after n trials was found to be

$$P = \frac{0.83}{1 + e^{-0.2n}}.$$

(a) Find the proportion of correct responses after $n = 10$ trials.
(b) Find the limiting proportion of correct responses as n approaches infinity.

44. In a typing class, the average number N of words per minute typed after t weeks of lessons was found to be

$$N = \frac{157}{1 + 5.4e^{-0.12t}}.$$

(a) Find the average number of words per minute after $t = 10$ weeks.
(b) Find the limiting number of words per minute as t approaches infinity.

45. In the Chapter 7 Application we introduced the following equation of the catenary for the Gateway Arch:

$$y \approx 757.71 - 127.71\left(\frac{e^{x/127.71} + e^{-x/127.71}}{2}\right).$$

Show that the height of the Gateway Arch is the same as the distance between its two legs.

46. Given the function

$$f(x) = \frac{2}{1 + e^{1/x}}$$

use a computer or graphics calculator to (a) sketch the graph of f, (b) find any horizontal asymptotes, and (c) find $\lim_{x\to 0} f(x)$ (if it exists).

7.2 Differentiation and Integration of Exponential Functions

Differentiation of exponential functions ▪ Integration of exponential functions

In Section 7.1 we claimed that the natural base e is the most convenient base for exponential functions. One reason for this claim is that the natural exponential function $f(x) = e^x$ is its own derivative. To prove this, consider the following.

$$\begin{aligned} f'(x) &= \lim_{\Delta x\to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x\to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x} \\ &= \lim_{\Delta x\to 0} \frac{e^x[e^{\Delta x} - 1]}{\Delta x} \end{aligned}$$

EXAMPLE 7 Finding the area of a region in the plane

Find the area of the region bounded by the graph of $f(x) = e^{-x}$ and the x-axis, for $0 \le x \le 1$.

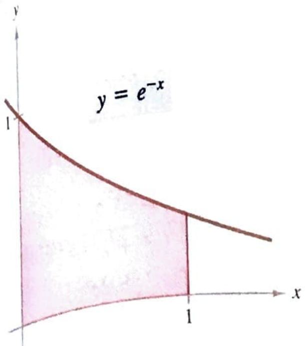


FIGURE 7.10

SOLUTION

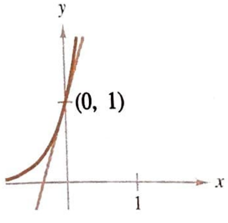
The region is shown in Figure 7.10, and its area is given by

$$\text{area} = \int_0^1 f(x)\,dx = \int_0^1 e^{-x}\,dx$$
$$= \Big[-e^{-x}\Big]_0^1$$
$$= -e^{-1} - (-1)$$
$$= 1 - \frac{1}{e} \approx 0.632.$$

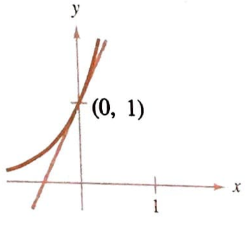
EXERCISES for Section 7.2

In Exercises 1–6, find the slope of the tangent line to the given exponential function at the point (0, 1).

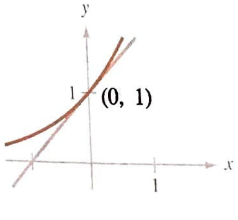
1. $y = e^{3x}$



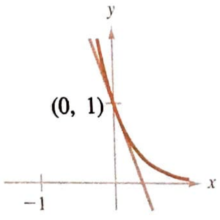
2. $y = e^{2x}$



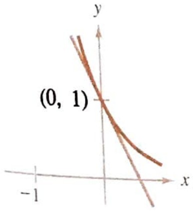
3. $y = e^{x}$



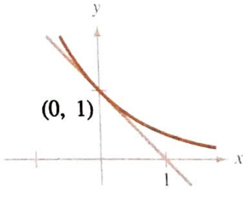
4. $y = e^{-3x}$



5. $y = e^{-2x}$



6. $y = e^{-x}$



In Exercises 7–24, find the derivative of the given function.

7. $y = e^{2x}$ **8.** $y = e^{1-x}$

9. $y = e^{-2x+x^2}$ **10.** $y = e^{-x^2}$

11. $f(x) = e^{1/x}$ **12.** $f(x) = e^{-1/x^2}$

13. $g(x) = e^{\sqrt{x}}$ **14.** $g(x) = e^{x^3}$

15. $f(x) = (x + 1)e^{3x}$ **16.** $y = x^2e^{-x}$

17. $f(x) = \dfrac{e^{x^2}}{x}$ **18.** $f(x) = \dfrac{e^{x/2}}{\sqrt{x}}$

19. $y = (e^{-x} + e^{x})^3$ **20.** $y = (1 - e^{-x})^2$

21. $f(x) = \dfrac{2}{e^x + e^{-x}}$ **22.** $f(x) = \dfrac{e^x - e^{-x}}{2}$

23. $y = xe^x - e^x$ **24.** $y = x^2e^x - 2xe^x + 2e^x$

In Exercises 25 and 26, use implicit differentiation to find dy/dx.

25. $xe^y - 10x + 3y = 0$ **26.** $e^{xy} + x^2 - y^2 = 10$

In Exercises 27–30, find the second derivative of the exponential function.

27. $f(x) = 2e^{3x} + 3e^{-2x}$ **28.** $f(x) = 5e^{-x} - 2e^{-5x}$

29. $g(x) = (1 + 2x)e^{4x}$ **30.** $g(x) = (3 + 2x)e^{-3x}$

In Exercises 31–34, find the extrema and the points of inflection (if any exist) and sketch the graph of the function.

31. $f(x) = \dfrac{2}{1 + e^{-x}}$ **32.** $f(x) = \dfrac{e^x - e^{-x}}{2}$

33. $f(x) = x^2e^{-x}$ **34.** $f(x) = xe^{-x}$

35. Find an equation of the line normal to the graph of $y = e^{-x}$ at $(0, 1)$.
36. Find the point on the graph of $y = e^{-x}$ where the normal line to the curve will pass through the origin.
37. Find the area of the largest rectangle that can be inscribed under the curve $y = e^{-x^2}$ in the first and second quadrants.
38. Find, to three decimal places, the value of x such that $e^{-x} = x$. [Use Newton's Method.]

In Exercises 39 and 40, use a graphing utility to graph the function. Then graph

$$P_1(x) = f(0) + f'(0)(x - 0)$$

and

$$P_2(x) = f(0) + f'(0)(x - 0) + \frac{1}{2}f''(0)(x - 0)^2$$

in the same viewing rectangle. Compare the values of f, P_1, and P_2 and their first derivatives at $x = 0$.

39. $f(x) = e^{x/2}$

40. $f(x) = e^{-x^2/2}$

41. Consider the function

$$f(x) = \frac{2}{1 + e^{1/x}}.$$

(a) Use a graphing utility to graph f.
(b) Write a short paragraph explaining why the graph has a horizontal asymptote $y = 1$ and why the function has a nonremovable discontinuity at $x = 0$.

42. The value V of an item t years after it is purchased is

$$V = 15{,}000e^{-0.6286t}, \qquad 0 \le t \le 10.$$

(a) Use a graphing utility to graph the function.
(b) Find the rate of change of V with respect to t when $t = 1$ and $t = 5$.
(c) Sketch the tangent line to the graph of the function when $t = 1$ and $t = 5$.

In Exercises 43–62, evaluate the integral.

43. $\int_0^1 e^{-2x}\,dx$

44. $\int_1^2 e^{1-x}\,dx$

45. $\int_0^2 (x^2 - 1)e^{x^3-3x+1}\,dx$

46. $\int x^2 e^{x^3}\,dx$

47. $\int \frac{e^{-x}}{(1 + e^{-x})^2}\,dx$

48. $\int \frac{e^{2x}}{(1 + e^{2x})^2}\,dx$

49. $\int xe^{ax^2}\,dx$

50. $\int_0^{\sqrt{2}} xe^{-(x^2/2)}\,dx$

51. $\int_1^3 \frac{e^{3/x}}{x^2}\,dx$

52. $\int (e^x - e^{-x})^2\,dx$

53. $\int e^{-x}(1 + e^{-x})^2\,dx$

54. $\int e^{2x}(1 - 3e^{2x})^{-2}\,dx$

55. $\int e^x\sqrt{1 - e^x}\,dx$

56. $\int e^x(e^x - e^{-x})\,dx$

57. $\int \frac{e^x + e^{-x}}{\sqrt{e^x - e^{-x}}}\,dx$

58. $\int \frac{2e^x - 2e^{-x}}{(e^x + e^{-x})^2}\,dx$

59. $\int \frac{5 - e^x}{e^{2x}}\,dx$

60. $\int \frac{e^{2x} + 2e^x + 1}{e^x}\,dx$

61. $\int_{-2}^0 (3^3 - 5^2)\,dx$

62. $\int (3 - x)e^{(3-x)^2}\,dx$

In Exercises 63 and 64, find a function f that satisfies the given conditions.

63. $f''(x) = \frac{1}{2}(e^x + e^{-x})$
$f(0) = 1, f'(0) = 0$

64. $f''(x) = x + e^{-2x}$
$f(0) = 3, f'(0) = -\frac{1}{2}$

In Exercises 65–68, find the area of the region bounded by the graphs of the given equations.

65. $y = e^x,\ y = 0,\ x = 0,\ x = 5$
66. $y = e^{-x},\ y = 0,\ x = a,\ x = b$
67. $y = xe^{-(x^2/2)},\ y = 0,\ x = 0,\ x = \sqrt{2}$
68. $y = e^{-2x} + 2,\ y = 0,\ x = 0,\ x = 2$

In Exercises 69 and 70, find the volume of the solid generated by revolving the region bounded by the graphs of the given equations about the x-axis.

69. $y = e^x,\ y = 0,\ x = 0,\ x = 1$
70. $y = e^{-x/2},\ y = 0,\ x = 0,\ x = 4$

71. Given $e^x \ge 1$ for $x \ge 0$, it follows that

$$\int_0^x e^t\,dt \ge \int_0^x 1\,dt.$$

Perform this integration to derive the inequality $e^x \ge 1 + x$ for $x \ge 0$.

72. Integrate each term of the following inequalities in a manner similar to that of Exercise 71 to obtain each succeeding inequality for $x \ge 0$. Then evaluate both sides of each inequality when $x = 1$.
(a) $e^x \ge 1 + x$
(b) $e^x \ge 1 + x + \frac{x^2}{2}$
(c) $e^x \ge 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$
(d) $e^x \ge 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$

Thus, in both cases, the slopes are reciprocals, as shown in Figure 7.18.

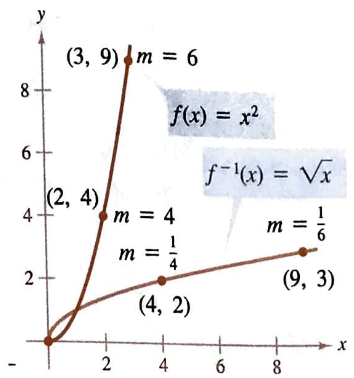


FIGURE 7.18

REMARK In Example 5, note that f^{-1} is not differentiable at (0, 0). This is consistent with Theorem 6.10 because the derivative of f at (0, 0) is zero.

EXERCISES for Section 7.3

In Exercises 1–8, (a) show that f and g are inverse functions by showing that $f(g(x)) = x$ and $g(f(x)) = x$, and (b) graph f and g on the same set of coordinate axes.

1. $f(x) = x^3$ $\quad g(x) = \sqrt[3]{x}$

2. $f(x) = x^{-1}$ $\quad g(x) = x^{-1}$

3. $f(x) = 5x + 1$ $\quad g(x) = \frac{1}{5}(x - 1)$

4. $f(x) = 3 - 4x$ $\quad g(x) = \frac{1}{4}(3 - x)$

5. $f(x) = \sqrt{x - 4}$ $\quad g(x) = x^2 + 4, x \ge 0$

6. $f(x) = 9 - x^2, x \ge 0$ $\quad g(x) = \sqrt{9 - x}$

7. $f(x) = 1 - x^3$ $\quad g(x) = \sqrt[3]{1 - x}$

8. $f(x) = x^{-2}, x > 0$ $\quad g(x) = x^{-1/2}, x > 0$

In Exercises 9–22, find the inverse of f. Then graph both f and f^{-1}.

9. $f(x) = 2x - 3$

10. $f(x) = 3x$

11. $f(x) = x^5$

12. $f(x) = x^3 + 1$

13. $f(x) = \sqrt{x}$

14. $f(x) = x^2, x \ge 0$

15. $f(x) = \sqrt{4 - x^2}, 0 \le x$

16. $f(x) = \sqrt{x^2 - 4}, x \ge 2$

17. $f(x) = \sqrt[3]{x - 1}$

18. $f(x) = 3\sqrt[5]{2x - 1}$

19. $f(x) = x^{2/3}, x \ge 0$

20. $f(x) = x^{3/5}$

21. $f(x) = \dfrac{x}{\sqrt{x^2 + 7}}$

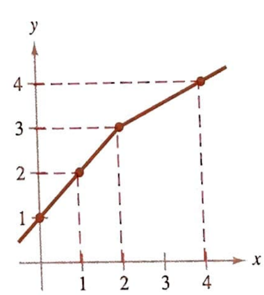
22. $f(x) = \dfrac{x + 2}{x}$

In Exercises 23 and 24, find the inverse function of f over the specified interval. Use a graphing utility to graph f and f^{-1} in the same viewing rectangle. Observe that the graph of f^{-1} is a reflection of the graph of f in the line $y = x$.

	Function	*Interval*
23.	$f(x) = \dfrac{x}{x^2 - 4}$	$-2 < x < 2$
24.	$f(x) = 2 - \dfrac{3}{x^2}$	$0 < x < 10$

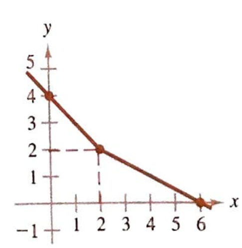
In Exercises 25 and 26, use the graph of the function f to complete the table and sketch the graph of f^{-1}.

25.



x	1	2	3	4
$f^{-1}(x)$				

26.



x	0	2	4
$f^{-1}(x)$			

In Exercises 27–30, use the functions

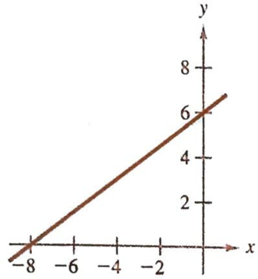
$$f(x) = \frac{1}{8}x - 3 \quad \text{and} \quad g(x) = x^3$$

to find the indicated value.

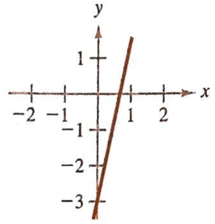
27. $(f^{-1} \circ g^{-1})(1)$ **28.** $(g^{-1} \circ f^{-1})(-3)$
29. $(f^{-1} \circ f^{-1})(6)$ **30.** $(g^{-1} \circ g^{-1})(-4)$

In Exercises 31–36, use the horizontal line test to determine whether the function is one-to-one on its entire domain and therefore has an inverse.

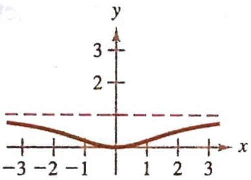
31. $f(x) = \frac{3}{4}x + 6$



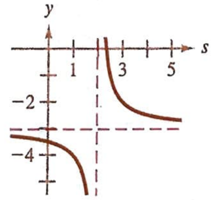
32. $f(x) = 5x - 3$



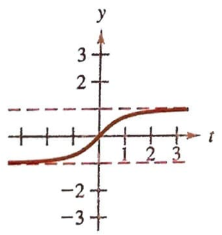
33. $f(x) = \dfrac{x^2}{x^2 + 4}$



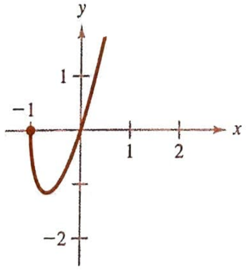
34. $h(s) = \dfrac{1}{s - 2} - 3$



35. $g(t) = \dfrac{t}{\sqrt{t^2 + 1}}$



36. $f(x) = 3x\sqrt{x + 1}$



In Exercises 37–40, use the derivative to determine whether the given function is strictly monotonic on its entire domain and therefore has an inverse.

37. $f(x) = (x + a)^3 + b$ **38.** $f(x) = \dfrac{x^4}{4} - 2x^2$
39. $f(x) = x^3 - 6x^2 + 12x$ **40.** $f(x) = 2 - x - x^3$

In Exercises 41–44, use a graphing utility to graph the function. From the graph, determine whether the function is one-to-one on its entire domain.

41. $h(s) = \dfrac{1}{s - 2} - 3$ **42.** $g(t) = \dfrac{1}{\sqrt{t^2 + 1}}$
43. $h(x) = |x + 4| - |x - 4|$
44. $g(x) = (x + 5)^3$

In Exercises 45–48, show that f is strictly monotonic on the given domain and therefore has an inverse on that domain.

Function	Domain
45. $f(x) = (x - 4)^2$	$[4, \infty)$
46. $f(x) = \lvert x + 2 \rvert$	$[-2, \infty)$
47. $f(x) = \dfrac{4}{x^2}$	$(0, \infty)$
48. $f(x) = x^3 - x$	$[1, \infty)$

In Exercises 49–52, show that the slopes of the graphs of f and f^{-1} are reciprocals at the given points.

Functions	Point
49. $f(x) = x^3$	$\left(\frac{1}{2}, \frac{1}{8}\right)$
$f^{-1}(x) = \sqrt[3]{x}$	$\left(\frac{1}{8}, \frac{1}{2}\right)$
50. $f(x) = 3 - 4x$	$(1, -1)$
$f^{-1}(x) = \dfrac{3 - x}{4}$	$(-1, 1)$
51. $f(x) = \sqrt{x - 4}$	$(5, 1)$
$f^{-1}(x) = x^2 + 4$	$(1, 5)$
52. $f(x) = \dfrac{1}{1 + x^2}$	$\left(1, \frac{1}{2}\right)$
$f^{-1}(x) = \sqrt{\dfrac{1 - x}{x}}$	$\left(\frac{1}{2}, 1\right)$

In Exercises 53 and 54, the derivative of the function has the same sign for all x in its domain, but the function is not strictly monotonic. Explain why.

53. $f(x) = \dfrac{1}{x}$ **54.** $f(x) = \dfrac{x}{x^2 - 4}$

55. Prove that if a function has an inverse, then the inverse is unique.

56. Prove that if f has an inverse, then $(f^{-1})^{-1} = f$.

57. Prove that a function has an inverse if and only if it is one-to-one.

58. Prove that if f and g are one-to-one functions, then $(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$.

Most calculators have both natural and common logarithmic keys (often denoted by $\ln x$ and $\log x$). Suppose, however, that you are asked to evaluate a logarithm to a base other than e or 10. In such cases, you can use the following change-of-base formula.

$$\log_a x = \frac{\log_b x}{\log_b a} \qquad \text{Change-of-base formula}$$

With this formula, we can use a calculator to evaluate expressions such as $\log_2 14$. That is,

$$\log_2 14 = \frac{\ln 14}{\ln 2} \approx \frac{2.63906}{0.693147} \approx 3.8074.$$

EXERCISES for Section 7.4

In Exercises 1–6, write the logarithmic equation as an exponential equation and vice versa.

1. (a) $2^3 = 8$ (b) $3^{-1} = \frac{1}{3}$
2. (a) $27^{2/3} = 9$ (b) $16^{3/4} = 8$
3. (a) $\log_{10} 0.01 = -2$ (b) $\log_{0.5} 8 = -3$
4. (a) $e^0 = 1$ (b) $e^2 = 7.389\ldots$
5. (a) $\ln 2 = 0.6931\ldots$ (b) $\ln 8.4 = 2.128\ldots$
6. (a) $\ln 0.5 = -0.6931\ldots$ (b) $49^{1/2} = 7$

In Exercises 7–14, solve for x (or b).

7. (a) $\log_{10} 1000 = x$ (b) $\log_{10} 0.1 = x$
8. (a) $\log_4 \frac{1}{64} = x$ (b) $\log_5 25 = x$
9. (a) $\log_3 x = -1$ (b) $\log_2 x = -4$
10. (a) $\log_b 27 = 3$ (b) $\log_b 125 = 3$
11. (a) $\log_{27} x = -\frac{2}{3}$ (b) $\ln e^x = 3$
12. (a) $e^{\ln x} = 4$ (b) $\ln x = 2$
13. (a) $x^2 - x = \log_5 25$
    (b) $3x + 5 = \log_2 64$
14. (a) $\log_3 x + \log_3 (x - 2) = 1$
    (b) $\log_{10} (x + 3) - \log_{10} x = 1$

In Exercises 15–20, sketch the graph of the function.

15. $f(x) = 3 \ln x$
16. $f(x) = -2 \ln x$
17. $f(x) = \ln 2x$
18. $f(x) = \ln |x|$
19. $f(x) = \ln (x - 1)$
20. $f(x) = 2 + \ln x$

In Exercises 21–24, show that the given functions are inverses of each other by sketching their graphs on the same coordinate axes.

21. $f(x) = e^{2x}$, $g(x) = \ln \sqrt{x}$
22. $f(x) = e^x - 1$, $g(x) = \ln (x + 1)$
23. $f(x) = e^{x-1}$, $g(x) = 1 + \ln x$
24. $f(x) = e^{x/3}$, $g(x) = \ln x^3$

In Exercises 25–30, apply the inverse properties of $\ln x$ and e^x to simplify the given expression.

25. $\ln e^{x^2}$
26. $\ln e^{2x-1}$
27. $e^{\ln (5x+2)}$
28. $-1 + \ln e^{2x}$
29. $e^{\ln \sqrt{x}}$
30. $-8 + e^{\ln x^3}$

In Exercises 31 and 32, use the properties of logarithms and the fact that $\ln 2 \approx 0.6931$ and $\ln 3 \approx 1.0986$ to approximate the given logarithm.

31. (a) $\ln 6$ (b) $\ln \frac{2}{3}$
    (c) $\ln 81$ (d) $\ln \sqrt{3}$
32. (a) $\ln 0.25$ (b) $\ln 24$
    (c) $\ln \sqrt[3]{12}$ (d) $\ln \frac{1}{72}$

In Exercises 33–42, use the properties of logarithms to write each as a sum, difference, or multiple of logarithms.

33. $\ln \frac{2}{3}$
34. $\ln (xyz)$
35. $\ln \frac{xy}{z}$
36. $\ln \sqrt{a - 1}$
37. $\ln \sqrt{2^3}$
38. $\ln \frac{1}{5}$
39. $\ln \left(\frac{x^2 - 1}{x^3}\right)^3$
40. $\ln 3e^2$
41. $\ln z(z - 1)^2$
42. $\ln \frac{1}{e}$

In Exercises 43–48, write each expression as a logarithm of a single quantity.

43. $\ln (x-2) - \ln (x+2)$
44. $3 \ln x + 2 \ln y - 4 \ln z$
45. $\frac{1}{3}[2 \ln (x+3) + \ln x - \ln (x^2-1)]$
46. $2[\ln x - \ln (x+1) - \ln (x-1)]$
47. $2 \ln 3 - \frac{1}{2} \ln (x^2+1)$
48. $\frac{3}{2}[\ln (x^2+1) - \ln (x+1) - \ln (x-1)]$

In Exercises 49–60, solve for x or t.

49. $e^{\ln x} = 4$
50. $e^{\ln x^2} - 9 = 0$
51. $\ln x = 0$
52. $2 \ln x = 4$
53. $e^{x+1} = 4$
54. $e^{-0.5x} = 0.075$
55. $500e^{-0.11t} = 600$
56. $e^{-0.0174t} = 0.5$
57. $5^{2x} = 15$
58. $2^{1-x} = 6$
59. $500(1.07)^t = 1000$
60. $1000\left(1 + \frac{0.07}{12}\right)^{12t} = 3000$

61. A deposit of \$1000 is made into a fund with an annual interest rate of 11%. Find the time for the investment to double if the interest is compounded
(a) annually. (b) monthly.
(c) daily. (d) continuously.

62. A deposit of \$1000 is made into a fund with an annual interest rate of $10\frac{1}{2}$%. Find the time for the investment to triple if the interest is compounded
(a) annually. (b) monthly.
(c) daily. (d) continuously.

63. Complete the following table for the time t necessary for P dollars to triple if interest is compounded continuously at the rate r.

r	2%	4%	6%	8%	10%	12%
t						

64. The demand function for a certain product is given by

$p = 500$

x	y	$\frac{\ln x}{\ln y}$	$\ln \frac{x}{y}$	$\ln x - \ln y$
1	2			
3	4			
10	5			
4	0.5			

66. There are 25 prime numbers less than 100. Number Theorem states that if $p(x)$ is the primes less than x, then the ratio of $p(x$ approaches 1 as x approaches infinity $x/\ln x$ for $x = 1000$, $x = 1{,}000{,}000$ $1{,}000{,}000{,}000$. Then compute the ratio $x/\ln x$ given that $p(1000) = 168$, $p(10^6$ and $p(10^9) = 50{,}847{,}478$.

In Exercises 67 and 68, show that $f = g$ computer or graphics calculator to sketch f and g on the same coordinate axes. (Ass

67. $f(x) = \ln \frac{x^2}{4}$
$g(x) = 2 \ln x - \ln 4$

68. $f(x) = \ln \sqrt{x(x^2+1)}$
$g(x) = \frac{1}{2}[\ln x + \ln (x^2+1)]$

In Exercises 69 and 70, evaluate the lo the change of base formula. Do each prob first time use common logarithms, and th use natural logarithms. Round your answ imal places.

69. (a) $\log_3 7$ (b) $\log_7 4$
(c) $\log_{1/2} 10$ (d) $\log_4 0.$
70. (a) $\log_9 0.4$ (b) $\log_2 0.$
(c) $\log_{15} 1250$ (d) $\log_{1/3}$

71. Prove that

$$\ln \frac{x}{y} = \ln x - \ln y.$$

72. Prove that $\ln x^y = y \ln x$.

EXAMPLE 10 Comparing variables and constants

(a) $\dfrac{d}{dx}[e^e] = 0$ — Constant Rule

(b) $\dfrac{d}{dx}[e^x] = e^x$ — Exponential Rule

(c) $\dfrac{d}{dx}[x^e] = ex^{e-1}$ — Power Rule

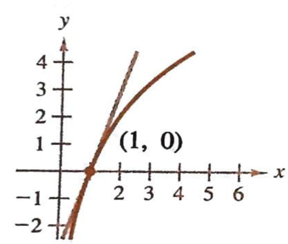
(d) — Logarithmic differentiation

$$y = x^x$$
$$\ln y = x \ln x$$
$$\frac{y'}{y} = x\left(\frac{1}{x}\right) + (\ln x)(1) = 1 + \ln x$$
$$y' = y(1 + \ln x) = x^x(1 + \ln x)$$

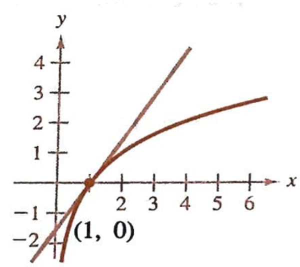
EXERCISES for Section 7.5

In Exercises 1–4, find the slope of the tangent line to the given logarithmic function at the point (1, 0).

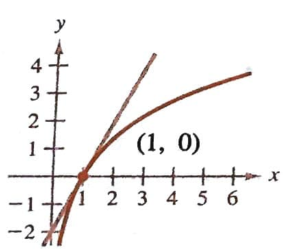
1. $y = \ln x^3$



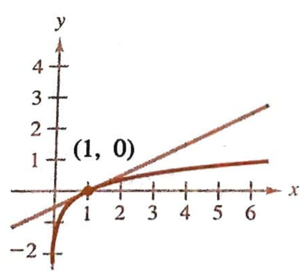
2. $y = \ln x^{3/2}$



3. $y = \ln x^2$



4. $y = \ln x^{1/2}$



In Exercises 5–38, find dy/dx.

5. $y = \ln x^2$
6. $y = \ln (x^2 + 3)$
7. $y = \ln \sqrt{x^4 - 4x}$
8. $y = \ln (1 - x)^{3/2}$
9. $y = (\ln x)^4$
10. $y = x \ln x$
11. $y = \ln (x\sqrt{x^2 - 1})$
12. $y = \ln \left(\dfrac{x}{x + 1}\right)$
13. $y = \ln \left(\dfrac{x}{x^2 + 1}\right)$
14. $y = \dfrac{\ln x}{x}$
15. $y = \dfrac{\ln x}{x^2}$
16. $y = \ln (\ln x)$
17. $y = \ln (\ln x^2)$
18. $y = \ln \sqrt{\dfrac{x - 1}{x + 1}}$
19. $y = \ln \sqrt{\dfrac{x + 1}{x - 1}}$
20. $y = \ln \sqrt{x^2 - 4}$
21. $y = \ln \left(\dfrac{\sqrt{4 + x^2}}{x}\right)$
22. $y = \ln (x + \sqrt{4 + x^2})$
23. $y = \dfrac{-\sqrt{x^2 + 1}}{x} + \ln (x + \sqrt{x^2 + 1})$
24. $y = \dfrac{-\sqrt{x^2 + 4}}{2x^2} - \dfrac{1}{4} \ln \left(\dfrac{2 + \sqrt{x^2 + 4}}{x}\right)$
25. $y = 4^x$
26. $y = 2^{-x}$
27. $y = 5^{x-2}$
28. $y = x(7^{-3x})$
29. $y = x^2 2^x$
30. $y = 2^{x^2} 3^{-x}$
31. $y = \log_3 x$
32. $y = \log_{10} 2x$
33. $y = \log_2 \left(\dfrac{x^2}{x - 1}\right)$
34. $y = \log_3 \left(\dfrac{x\sqrt{x - 1}}{2}\right)$
35. $y = \log_5 \sqrt{x^2 - 1}$
36. $y = \log_{10} \left(\dfrac{x^2 - 1}{x}\right)$
37. $y = \ln |x^2 - 1|$
38. $y = \ln \left|\dfrac{x + 5}{x}\right|$

In Exercises 39–48, find dy/dx using logarithmic differentiation.

39. $y = x\sqrt{x^2 - 1}$
40. $y = \sqrt{(x - 1)(x - 2)(x - 3)}$
41. $y = \dfrac{x^2\sqrt{3x - 2}}{(x - 1)^2}$
42. $y = \sqrt[3]{\dfrac{x^2 + 1}{x^2 - 1}}$
43. $y = \dfrac{x(x - 1)^{3/2}}{\sqrt{x + 1}}$
44. $y = \dfrac{(x + 1)(x + 2)}{(x - 1)(x - 2)}$
45. $y = x^{2/x}$
46. $y = x^{x-1}$
47. $y = (x - 2)^{x+1}$
48. $y = (1 + x)^{1/x}$

In Exercises 49 and 50, show that the given function is a solution to the differential equation.

Function	*Differential equation*
49. $y = 2 \ln x + 3$	$x(y'') + y' = 0$
50. $y = x \ln x - 4x$	$x + y - xy' = 0$

In Exercises 51 and 52, find dy/dx by using implicit differentiation.

51. $x^2 - 3 \ln y + y^2 = 10$

52. $\ln xy + 5x = 30$

In Exercises 53 and 54, find an equation of the tangent line to the graph of the equation at the given point.

Equation	*Point*
53. $y = 3x^2 - \ln x$	$(1, 3)$
54. $x^2 + \ln(x + 1) + y^2 = 4$	$(0, 2)$

In Exercises 55–60, find any relative extrema and inflection points, and sketch the graph of the function.

55. $y = \dfrac{x^2}{2} - \ln x$

56. $y = x - \ln x$

57. $y = x(\ln x)$

58. $y = \dfrac{\ln x}{x}$

59. $y = \dfrac{x}{\ln x}$

60. $y = x^2(\ln x)$

In Exercises 61 and 62, use Newton's Method to approximate, to three decimal places, the x-coordinate of the point of intersection of the graphs of the two equations.

61. $y = \ln x$
$y = -x$

62. $y = \ln x$
$y = 3 - x$

63. Let $L(x) = \int_1^x (1/t)\, dt$ for all $x > 0$.

(a) Find $L(1)$.

(b) Find $L'(x)$ and $L'(1)$.

(c) Use the Trapezoidal Rule to approximate the value of x (to three decimal places) for which $L(x) = 1$.

(d) Show that L is an increasing function on $(0, \infty)$.

(e) Prove that $L(x_1x_2) = L(x_1) + L(x_2)$ for $x_1 > 0$ and $x_2 > 0$.

64. Show that

$$f(x) = \frac{\ln x^n}{x}$$

is a decreasing function for $x > e$ and $n > 0$.

65. A person walking along a dock drags a boat by a 10-foot rope. The boat travels along a path known as a *tractrix* (see figure). The equation of the path is

$$y = 10 \ln\left(\frac{10 + \sqrt{100 - x^2}}{x}\right) - \sqrt{100 - x^2}.$$

What is the slope of this path at the following x-values?

(a) $x = 10$ (b) $x = 5$

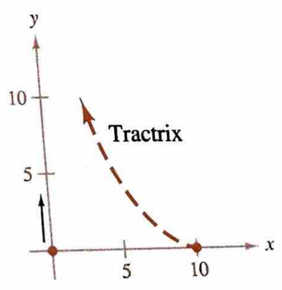


FIGURE FOR 65

7.6 Logarithmic Functions and Integration

The Log Rule for integration

The differentiation rules

$$\frac{d}{dx}[\ln|x|] = \frac{1}{x} \quad \text{and} \quad \frac{d}{dx}[\ln|u|] = \frac{u'}{u}$$

allow us to patch up the hole in our General Power Rule for integration. Recall from Section 5.5 that

$$\int u^n\, du = \frac{u^{n+1}}{n+1} + C$$

provided $n \neq -1$. Having the differentiation formulas for logarithmic functions, we are now in a position to evaluate $\int u^n\, du$ for $n = -1$, as stated in the following theorem.