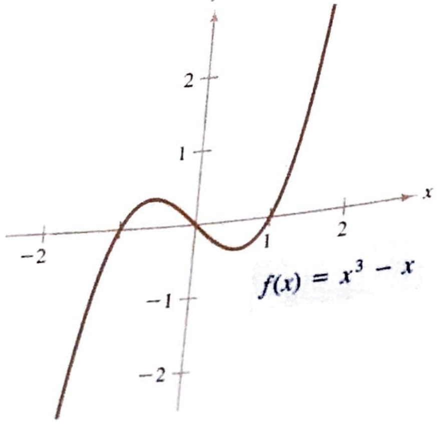
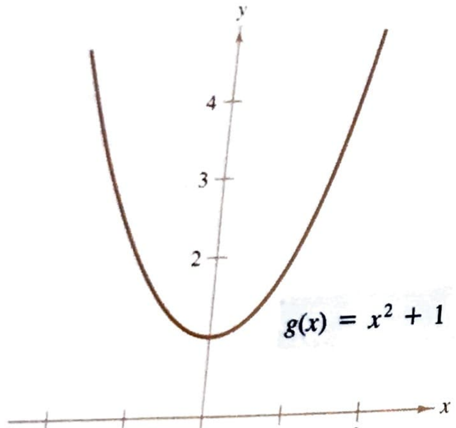
Determine whether ... case find the zeros of the func...

(a) $f(x) = x^3 - x$ (b) $g(x) = x^2 + 1$



(a) Odd function



(b) Even function

FIGURE 1.55

SOLUTION

(a) This function is odd since

$$f(-x) = (-x)^3 - (-x) = -x^3 + x = -(x^3 - x) = -f(x).$$

The zeros of f are found as follows.

$$x^3 - x = 0 \qquad \text{Let } f(x) = 0$$

$$x(x^2 - 1) = x(x - 1)(x + 1) = 0 \qquad \text{Factor}$$

$$x = 0,\ 1,\ -1 \qquad \text{Zeros of } f$$

(b) This function is even since

$$g(-x) = (-x)^2 + 1 = x^2 + 1 = g(x).$$

It has no zeros since $x^2 + 1$ is positive for all x. (See Figure 1.55.)

REMARK Each of the functions in Example 5 is either even or odd. However, so[me] functions such as

$$f(x) = x^2 + x + 1$$

are neither even nor odd.

EXERCISES for Section 1.5

1. Given $f(x) = 2x - 3$, find the following.
   (a) $f(0)$ (b) $f(-3)$
   (c) $f(b)$ (d) $f(x - 1)$
2. Given $f(x) = x^2 - 2x + 2$, find the following.
   (a) $f\left(\frac{1}{2}\right)$ (b) $f(-1)$
   (c) $f(c)$ (d) $f(x + \Delta x)$
3. Given $f(x) = \sqrt{x + 3}$, find the following.
   (a) $f(-2)$ (b) $f(6)$
   (c) $f(c)$ (d) $f(x + \Delta x)$
4. Given $f(x) = 1/\sqrt{x}$, find the following.
   (a) $f(2)$ (b) $f\left(\frac{1}{4}\right)$
   (c) $f(x + \Delta x)$ (d) $f(x + \Delta x) - f(x)$
5. Given $f(x) = |x|/x$, find the following.
   (a) $f(2)$ (b) $f(-2)$
   (c) $f(x^2)$ (d) $f(x - 1)$
6. Given $f(x) = |x| + 4$, find the following.
   (a) $f(2)$ (b) $f(-2)$
   (c) $f(x^2)$ (d) $f(x + \Delta x) - f(x)$

7. Given $f(x) = x^2 - x + 1$, find
$$\frac{f(2 + \Delta x) - f(2)}{\Delta x}.$$

8. Given $f(x) = 1/x$, find
$$\frac{f(1 + \Delta x) - f(1)}{\Delta x}.$$

9. Given $f(x) = x^3$, find
$$\frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

10. Given $f(x) = 3x - 1$, find
$$\frac{f(x) - f(1)}{x - 1}.$$

11. Given $f(x) = 1/\sqrt{x - 1}$, find
$$\frac{f(x) - f(2)}{x - 2}.$$

12. Given $f(x) = x^3 - x$, find
$$\frac{f(x) - f(1)}{x - 1}.$$

In Exercises 13–22, find the domain and range of the given function, and sketch its graph.

13. $f(x) = 4 - x$

14. $f(x) = \frac{1}{3}x$

15. $f(x) = 4 - x^2$

16. $g(x) = \frac{4}{x}$

17. $h(x) = \sqrt{x - 1}$

18. $f(x) = \frac{1}{2}x^3 + 2$

19. $f(x) = \sqrt{9 - x^2}$

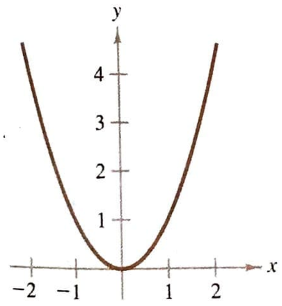
20. $h(x) = \sqrt{25 - x^2}$

21. $f(x) = |x - 2|$

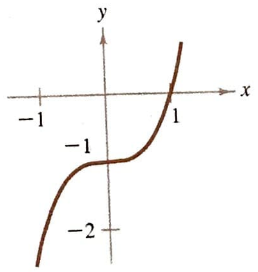
22. $f(x) = \frac{|x|}{x}$

In Exercises 23–28, use the vertical line test to determine whether y is a function of x.

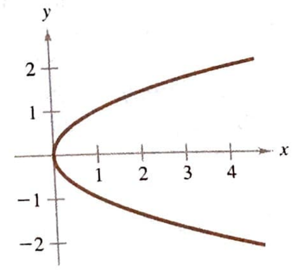
23. $y = x^2$



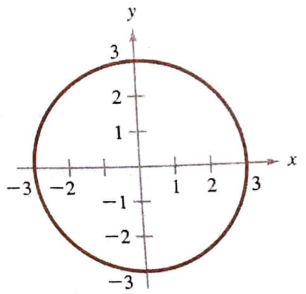
24. $y = x^3 - 1$



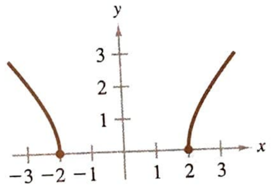
25. $x - y^2 = 0$



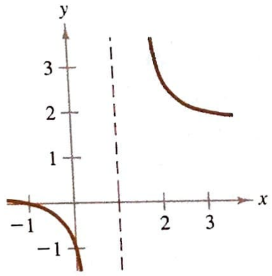
26. $x^2 + y^2 = 9$



27. $\sqrt{x^2 - 4} - y = 0$



28. $x - xy + y + 1 = 0$



In Exercises 29–36, determine whether y is a function of x.

29. $x^2 + y^2 = 4$

30. $x = y^2$

31. $x^2 + y = 4$

32. $x + y^2 = 4$

33. $2x + 3y = 4$

34. $x^2 + y^2 - 4y = 0$

35. $y^2 = x^2 - 1$

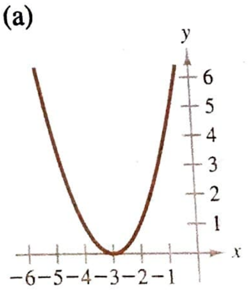
36. $x^2y - x^2 + 4y = 0$

37. Use the graph of $f(x) = \sqrt{x}$ to sketch the graph of each of the following.
(a) $y = \sqrt{x} + 2$
(b) $y = -\sqrt{x}$
(c) $y = \sqrt{x} - 2$
(d) $y = \sqrt{x + 3}$
(e) $y = \sqrt{x - 4}$
(f) $y = 2\sqrt{x}$

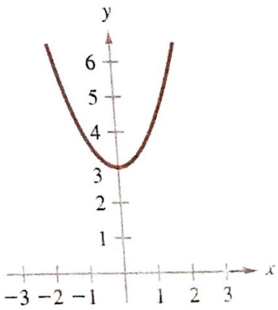
38. Use the graph of $f(x) = 1/x$ to sketch the graph of each of the following.
(a) $y = \frac{1}{x} - 1$
(b) $y = \frac{1}{x + 1}$
(c) $y = \frac{1}{x - 1}$
(d) $y = -\frac{1}{x}$
(e) $y = \frac{4}{x}$
(f) $y = -\frac{1}{x} + 2$

39. Use the graph of $f(x) = x^2$ to determine a formula for the indicated function.

(a)



(b)



SOLUTION

Since

$$\lim_{x\to 3} (2x^2 - 10) = 2(3^2) - 10 = 8$$

we have

$$\lim_{x\to 3} \sqrt[3]{2x^2 - 10} = \sqrt[3]{8} = 2.$$

EXERCISES for Section 2.1

In Exercises 1–6, complete the table and use the result to estimate the given limit.

1. $\displaystyle\lim_{x\to 2} \frac{x-2}{x^2 - x - 2}$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$						

2. $\displaystyle\lim_{x\to 2} \frac{x-2}{x^2 - 4}$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$						

3. $\displaystyle\lim_{x\to 0} \frac{\sqrt{x+3} - \sqrt{3}}{x}$

x	−0.1	−0.01	−0.001
$f(x)$			

x	0.001	0.01	0.1
$f(x)$			

4. $\displaystyle\lim_{x\to -3} \frac{\sqrt{1-x} - 2}{x+3}$

x	−3.1	−3.01	−3.001
$f(x)$			

x	−2.999	−2.99	−2.9
$f(x)$			

5. $\displaystyle\lim_{x\to 3} \frac{[1/(x+1)] - (1/4)}{x-3}$

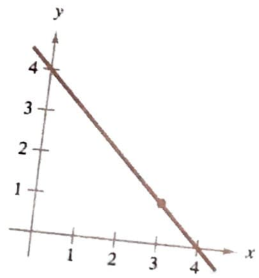
x	2.9	2.99	2.999	3.001	3.01	3.1
$f(x)$						

6. $\displaystyle\lim_{x\to 4} \frac{[x/(x+1)] - (4/5)}{x-4}$

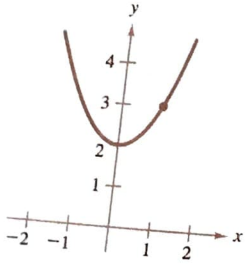
x	3.9	3.99	3.999	4.001	4.01	4.1
$f(x)$						

In Exercises 7–12, use the given graph to find the limit (if it exists).

7. $\lim_{x\to 3} (4 - x)$

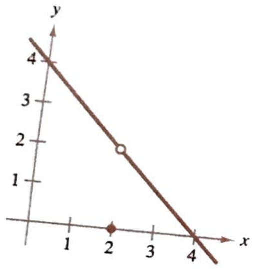


8. $\lim_{x\to 1} (x^2 + 2)$



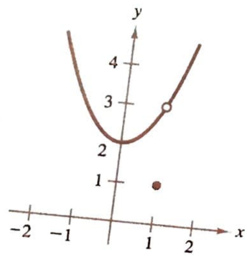
9. $\lim_{x\to 2} f(x)$

$$f(x) = \begin{cases} 4 - x, & x \neq 2 \\ 0, & x = 2 \end{cases}$$

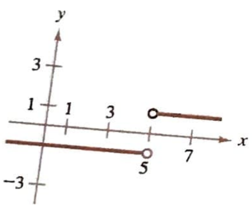


10. $\lim_{x\to 1} f(x)$

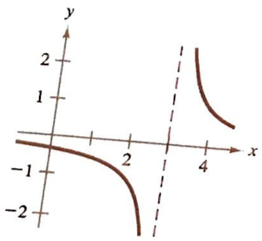
$$f(x) = \begin{cases} x^2 + 2, & x \neq 1 \\ 1, & x = 1 \end{cases}$$



11. $\lim_{x\to 5} \dfrac{|x - 5|}{x - 5}$



12. $\lim_{x\to 3} \dfrac{1}{x - 3}$



In Exercises 13–26, find the indicated limit.

13. $\lim_{x\to 2} x^2$

14. $\lim_{x\to -3} (3x + 2)$

15. $\lim_{x\to 0} (2x - 1)$

16. $\lim_{x\to 1} (-x^2 + 1)$

17. $\lim_{x\to 2} (-x^2 + x - 2)$

18. $\lim_{x\to 1} (3x^3 - 2x^2 + 4)$

19. $\lim_{x\to 3} \sqrt{x + 1}$

20. $\lim_{x\to 4} \sqrt[3]{x + 4}$

21. $\lim_{x\to -4} (x + 3)^2$

22. $\lim_{x\to 0} (2x - 1)^3$

23. $\lim_{x\to 2} \dfrac{1}{x}$

24. $\lim_{x\to -3} \dfrac{2}{x + 2}$

25. $\lim_{x\to -1} \dfrac{x^2 + 1}{x}$

26. $\lim_{x\to 3} \dfrac{\sqrt{x + 1}}{x - 4}$

27. If $\lim_{x\to c} f(x) = 2$ and $\lim_{x\to c} g(x) = 3$, find the following.

(a) $\lim_{x\to c} [5g(x)]$ (b) $\lim_{x\to c} [f(x) + g(x)]$

(c) $\lim_{x\to c} [f(x)g(x)]$ (d) $\lim_{x\to c} \dfrac{f(x)}{g(x)}$

28. If $\lim_{x\to c} f(x) = \frac{3}{2}$ and $\lim_{x\to c} g(x) = \frac{1}{2}$, find the following.

(a) $\lim_{x\to c} [4f(x)]$ (b) $\lim_{x\to c} [f(x) + g(x)]$

(c) $\lim_{x\to c} [f(x)g(x)]$ (d) $\lim_{x\to c} \dfrac{f(x)}{g(x)}$

29. If $\lim_{x\to c} f(x) = 4$, find the following.

(a) $\lim_{x\to c} [f(x)]^3$ (b) $\lim_{x\to c} \sqrt{f(x)}$

(c) $\lim_{x\to c} [3f(x)]$ (d) $\lim_{x\to c} [f(x)]^{3/2}$

30. If $\lim_{x\to c} f(x) = 27$, find the following.

(a) $\lim_{x\to c} \sqrt[3]{f(x)}$ (b) $\lim_{x\to c} \dfrac{f(x)}{18}$

(c) $\lim_{x\to c} [f(x)]^2$ (d) $\lim_{x\to c} [f(x)]^{2/3}$

In Exercises 31 and 32, use a computer or graphics calculator to sketch the graph of the function f and find the specified limit (if it exists).

31. $f(x) = \dfrac{\sqrt{x + 5} - 3}{x - 4}$, $\lim_{x\to 4} f(x)$

32. $f(x) = \dfrac{x - 3}{x^2 - 4x + 3}$, $\lim_{x\to 3} f(x)$

33. Write a computer program or use a spreadsheet to approximate $\lim_{x\to 4} f(x)$ where

$$f(x) = \frac{x^2 - x - 12}{x - 4}.$$

[Hint: Generate two columns whose entries form the ordered pairs $(4 \pm [0.1]^n, f(4 \pm [0.1]^n))$ for $n = 1$, 2, 3, and 4.]

34. If $f(2) = 4$, can we conclude anything about

$$\lim_{x\to 2} f(x)?$$

Give reasons for your answer.

35. If

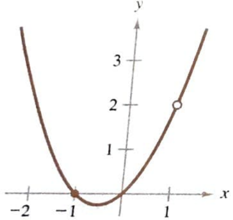
$$\lim_{x\to 2} f(x) = 4,$$

can we conclude anything about $f(2)$? Give reasons for your answer.

36. Find two functions f and g such that

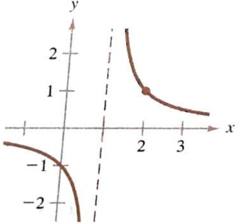
$$\lim_{x\to 0} f(x) \quad \text{and} \quad \lim_{x\to 0} g(x)$$

do not exist, but $\lim_{x\to 0} [f(x) + g(x)]$ does exist.



(a) $\lim_{x \to 1} g(x)$

(b) $\lim_{x \to -1} g(x)$



(a) $\lim_{x \to 1} f(x)$

(b) $\lim_{x \to 2} f(x)$

In Exercises 5–22, find the limit (if it exists).

5. $\lim_{x \to -1} \dfrac{x^2 - 1}{x + 1}$

6. $\lim_{x \to -1} \dfrac{2x^2 - x - 3}{x + 1}$

7. $\lim_{x \to 3} \dfrac{x - 3}{x^2 - 9}$

8. $\lim_{x \to -1} \dfrac{x^3 + 1}{x + 1}$

9. $\lim_{x \to -2} \dfrac{x^3 + 8}{x + 2}$

10. $\lim_{\Delta x \to 0} \dfrac{(x + \Delta x)^2 - x^2}{\Delta x}$

11. $\lim_{\Delta x \to 0} \dfrac{2(x + \Delta x) - 2x}{\Delta x}$

12. $\lim_{\Delta x \to 0} \dfrac{(x + \Delta x)^3 - x^3}{\Delta x}$

13. $\lim_{\Delta x \to 0} \dfrac{(x + \Delta x)^2 - 2(x + \Delta x) + 1 - (x^2 - 2x + 1)}{\Delta x}$

14. $\lim_{\Delta x \to 0} \dfrac{(1 + \Delta x)^3 - 1}{\Delta x}$

15. $\lim_{x \to 5} \dfrac{x - 5}{x^2 - 25}$

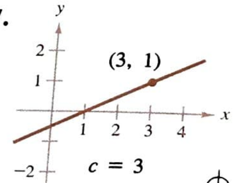
16. $\lim_{x \to 2} \dfrac{2 - x}{x^2 - 4}$

26. $\lim_{x \to 2}$ ... $x - 2$

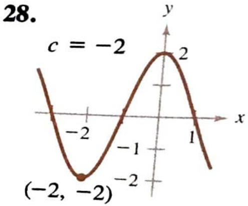
In Exercises 27–32, use the graph to determine the following visually.

(a) $\lim_{x \to c^+} f(x)$ (b) $\lim_{x \to c^-} f(x)$ (c) $\lim_{x \to c} f(x)$

27.



28.



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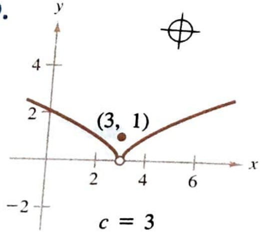

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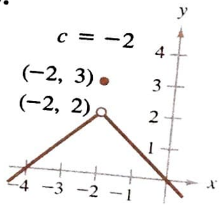

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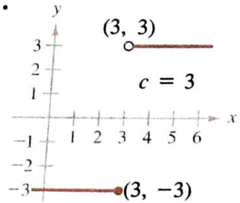

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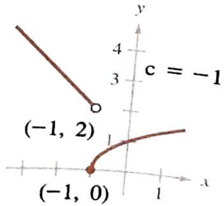


In Exercises 33–50, find the limit (if it exists).

33. $\lim\limits_{x\to 5^+} \dfrac{x-5}{x^2-25}$

34. $\lim\limits_{x\to 2^+} \dfrac{2-x}{x^2-4}$

35. $\lim\limits_{x\to 2^+} \dfrac{x}{\sqrt{x^2-4}}$

36. $\lim\limits_{x\to 4^-} \dfrac{\sqrt{x}-2}{x-4}$

37. $\lim\limits_{\Delta x\to 0^+} \dfrac{2(x+\Delta x)-2x}{\Delta x}$

38. $\lim\limits_{\Delta x\to 0^+} \dfrac{(x+\Delta x)^2+(x+\Delta x)-(x^2+x)}{\Delta x}$

39. $\lim\limits_{\Delta x\to 0^+} \dfrac{\dfrac{1}{x+\Delta x}-\dfrac{1}{x}}{\Delta x}$

40. $\lim\limits_{x\to 1^-} \dfrac{x^2-2x+1}{x-1}$

41. $\lim\limits_{x\to 0} \dfrac{|x|}{x}$

42. $\lim\limits_{x\to 2} \dfrac{|x-2|}{x-2}$

43. $\lim\limits_{x\to 3} f(x), \quad f(x) = \begin{cases} \dfrac{x+2}{2}, & x \le 3 \\ \dfrac{12-2x}{3}, & x > 3 \end{cases}$

44. $\lim\limits_{x\to 2} f(x), \quad f(x) = \begin{cases} x^2-4x+6, & x < 2 \\ -x^2+4x-2, & x \ge 2 \end{cases}$

45. $\lim\limits_{x\to 1} f(x), \quad f(x) = \begin{cases} x^3+1, & x < 1 \\ x+1, & x \ge 1 \end{cases}$

46. $\lim\limits_{x\to 1} f(x), \quad f(x) = \begin{cases} x, & x \le 1 \\ 1-x, & x > 1 \end{cases}$

47. $\lim\limits_{x\to 3^-} 2[\![x-3]\!]$

48. $\lim\limits_{x\to 1^+} [\![2x]\!]$

49. $\lim\limits_{x\to 1}\left(\left[\!\!\left[\dfrac{x}{4}\right]\!\!\right]+x\right)$

50. $\lim\limits_{x\to 2} \left[\!\!\left[\dfrac{x-1}{2}\right]\!\!\right]$

In Exercises 51 and 52, use the position function $s(t) = -16t^2 + 25$ giving the height (in feet) of a free-falling object. As discussed in the Chapter 2 Application, the velocity at time $t = a$ is given by

$$\lim_{t\to a} \frac{s(a)-s(t)}{a-t}.$$

51. Find the velocity when $t = 0.5$ second.

52. Find the velocity when $t = 1.1$ seconds.

In Exercises 53 and 54, use a graphing utility to graph the function and estimate the limit (if it exists). What is the domain of the function? Can you detect a possible danger in determining the domain of a function solely by analyzing the graph generated by a graphing utility? Write a short paragraph about the importance of examining a function analytically as well as graphically.

53. $f(x) = \dfrac{x-9}{\sqrt{x}-3}$

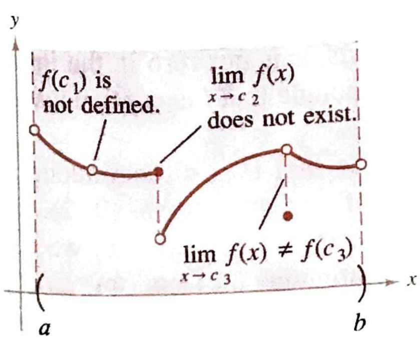
$\lim\limits_{x\to 9} f(x)$

54. $f(x) = \dfrac{x-3}{x^2-9}$

$\lim\limits_{x\to 3} f(x)$

2.3 Continuity

Continuity at a point ▪ Continuity on an open interval ▪ Continuity on a closed interval ▪ Properties of continuity ▪ Intermediate Value Theorem



Three Points of Discontinuity

FIGURE 2.14

In mathematics the term *continuous* has much the same meaning as it does in our everyday usage. To say that a function is continuous at $x = c$ means that there is no interruption in the graph of f at c. That is, its graph is unbroken at c and there are no holes, jumps, or gaps. For example, Figure 2.14 identifies three values of x at which the graph of f is not continuous. At all other points of the interval (a, b), the graph of f is uninterrupted and we say it is **continuous** at such points. Thus, it appears that the continuity of a function at $x = c$ can be destroyed by any one of the following conditions:

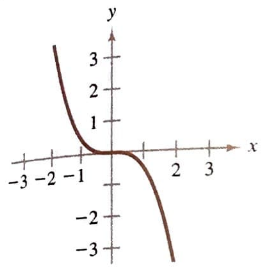
1. The function is not defined at $x = c$.
2. The limit of $f(x)$ does not exist at $x = c$.
3. The limit of $f(x)$ exists at $x = c$, but is not equal to $f(c)$.

This brings us to the following definition.

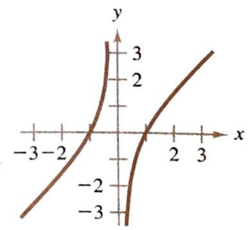
EXERCISES for Section 2.3

In Exercises 1–6, find the points of discontinuity (if any).

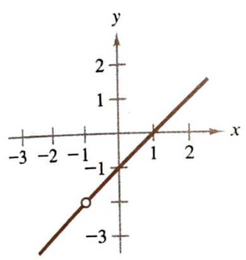
1. $f(x) = -\dfrac{x^3}{2}$



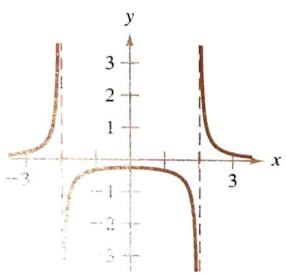
2. $f(x) = \dfrac{x^2 - 1}{x}$



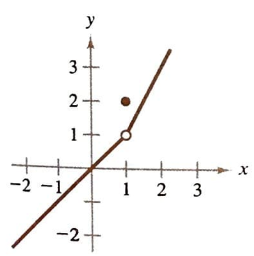
3. $f(x) = \dfrac{x^2 - 1}{x + 1}$



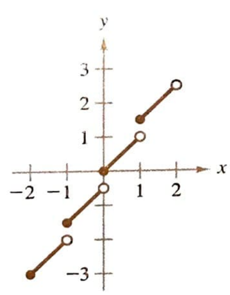
4. $f(x) = \dfrac{1}{x^2 - 4}$



5. $f(x) = \begin{cases} x, & x < 1 \\ 2, & x = 1 \\ 2x - 1, & x > 1 \end{cases}$



6. $f(x) = \dfrac{[\![x]\!]}{2} + x$



In Exercises 7–24, find the discontinuities (if any) for the given function. Which of the discontinuities are removable?

7. $f(x) = x^2 - 2x + 1$

8. $f(x) = \dfrac{1}{x^2 + 1}$

9. $f(x) = \dfrac{1}{x - 1}$

10. $f(x) = \dfrac{x}{x^2 - 1}$

11. $f(x) = \dfrac{x}{x^2 + 1}$

12. $f(x) = \dfrac{x - 3}{x^2 - 9}$

13. $f(x) = \dfrac{x + 2}{x^2 - 3x - 10}$

14. $f(x) = \dfrac{x - 1}{x^2 + x - 2}$

15. $f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$

16. $f(x) = \begin{cases} -2x + 3, & x < 1 \\ x^2, & x \ge 1 \end{cases}$

17. $f(x) = \begin{cases} \dfrac{x}{2} + 1, & x \le 2 \\ 3 - x, & x > 2 \end{cases}$

18. $f(x) = \begin{cases} -2x, & x \le 2 \\ x^2 - 4x + 1, & x > 2 \end{cases}$

19. $f(x) = \dfrac{|x + 2|}{x + 2}$

20. $f(x) = \dfrac{|x - 3|}{x - 3}$

21. $f(x) = \begin{cases} |x - 2| + 3, & x < 0 \\ x + 5, & x \ge 0 \end{cases}$

22. $f(x) = \begin{cases} 3 + x, & x \le 2 \\ x^2 + 1, & x > 2 \end{cases}$

23. $f(x) = [\![x - 1]\!]$

24. $f(x) = x - [\![x]\!]$

In Exercises 25–30, discuss the continuity of the composite function $h(x) = f(g(x))$.

25. $f(x) = x^2$, $g(x) = x - 1$

26. $f(x) = \dfrac{1}{\sqrt{x}}$, $g(x) = x - 1$

27. $f(x) = \dfrac{1}{x - 1}$, $g(x) = x^2 + 5$

28. $f(x) = \sqrt{x}$, $g(x) = x^2$

29. $f(x) = \dfrac{1}{x}$, $g(x) = \dfrac{1}{x - 1}$

30. $f(x) = \dfrac{1}{\sqrt{x}}$, $g(x) = \dfrac{1}{x}$

In Exercises 31–34, sketch the graph of the given function to determine any points of discontinuity.

31. $f(x) = \dfrac{x^2 - 16}{x - 4}$

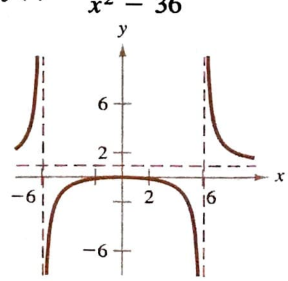
32. $f(x) = \dfrac{x^3 - 8}{x - 2}$

33. $f(x) = [\![x]\!] - x$

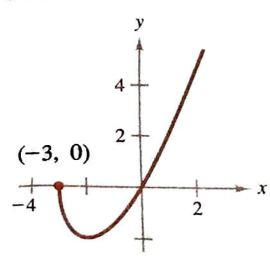
34. $f(x) = \dfrac{|x^2 - 1|}{x}$

In Exercises 35–38, find the interval(s) for which the function is continuous.

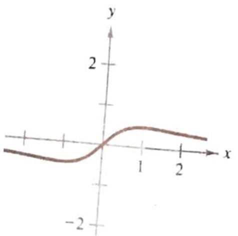
35. $f(x) = \dfrac{x^2}{x^2 - 36}$



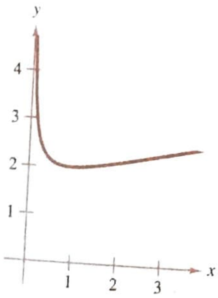
36. $f(x) = x\sqrt{x + 3}$



37. $f(x) = \dfrac{x}{x^2 + 1}$



38. $f(x) = \dfrac{x + 1}{\sqrt{x}}$



In Exercises 39 and 40, prove that the given function has a zero in the indicated interval.

Function	Interval
39. $f(x) = x^2 - 4x + 3$	$[2, 4]$
40. $f(x) = x^3 + 3x - 2$	$[0, 1]$

In Exercises 41 and 42, use the Intermediate Value Theorem to approximate the zero of the given function in the interval $[0, 1]$. (a) Begin by locating the zero in a subinterval of length 0.1. (b) Refine your approximation by locating the zero in a subinterval of length 0.01.

41. $f(x) = x^3 + x - 1$

42. $f(x) = x^3 + 3x - 2$

In Exercises 43–46, verify the applicability of the Intermediate Value Theorem in the indicated interval and find the value of c guaranteed by the theorem.

43. $f(x) = x^2 + x - 1$, $[0, 5]$, $f(c) = 11$

44. $f(x) = x^2 - 6x + 8$, $[0, 3]$, $f(c) = 0$

45. $f(x) = x^3 - x^2 + x - 2$, $[0, 3]$, $f(c) = 4$

46. $f(x) = \dfrac{x^2 + x}{x - 1}$, $\left[\dfrac{5}{2}, 4\right]$, $f(c) = 6$

47. Determine the constant a so that the following function is continuous on the entire real line.

$$f(x) = \begin{cases} x^3, & x \le 2 \\ ax^2, & x > 2 \end{cases}$$

48. Determine the constants a and b so that the following function is continuous on the entire real line.

$$f(x) = \begin{cases} 2, & x \le -1 \\ ax + b, & -1 < x < 3 \\ -2, & x \ge 3 \end{cases}$$

49. Is the function $f(x) = \sqrt{1 - x^2}$ continuous at $x = 1$? Give the reason for your answer.

50. A union contract guarantees a 9 percent annual salary increase for five years. For an initial salary of \$28,500, the salary S is given by

$$S = 28{,}500(1.09)^{[\![t]\!]}$$

where $t = 0$ corresponds to 1985. Sketch a graph of this function and discuss its continuity.

51. A dial-direct long distance call between two cities costs \$1.04 for the first two minutes and \$0.36 for each additional minute or fraction thereof. Use the greatest integer function to write the cost C of a call in terms of the time t (in minutes). Sketch a graph of this function and discuss its continuity.

52. The number of units in inventory in a small company is given by

$$N(t) = 25\left(2\left[\!\left[\frac{t + 2}{2}\right]\!\right] - t\right)$$

where t is the time in months. Sketch the graph of this function and discuss its continuity. How often must this company replenish its inventory?

53. Use a computer or graphics calculator to sketch the graph of the function f and determine whether it is continuous on the entire real line.

$$f(x) = \begin{cases} 2x - 4, & x \le 3 \\ x^2 - 2x, & x > 3 \end{cases}$$

54. At 8:00 A.M. on Saturday a man begins running up the side of a mountain to his weekend campsite. On Sunday at 8:00 A.M. he runs back down the mountain. It takes him 20 minutes to run up, but only 10 minutes to run down. At some point on the way down he realizes that he passed the same place at the exact same time on Saturday. Prove that he is correct. [Hint: Let $s(t)$ and $r(t)$ be the position functions for the run up and down, respectively, and apply the Intermediate Value Theorem to the function $f(t) = s(t) - r(t)$.]

55. Prove Theorem 2.9.

56. Prove that if f is continuous and has no zeros on $[a, b]$, then either

$$f(x) > 0 \quad \text{for all } x \text{ in } [a, b]$$

or

$$f(x) < 0 \quad \text{for all } x \text{ in } [a, b].$$

57. The Dirichlet function f is defined as follows.

$$f(x) = \begin{cases} 0, & x \text{ is rational} \\ 1, & x \text{ is irrational} \end{cases}$$

Show that this function is discontinuous at every real number.

SOLUTION

(a) Since $\lim_{x\to 0} (1) = 1$ and $\lim_{x\to 0} (1/x^2) = \infty$, we can apply Property 1 of Theorem 2.13 to conclude that

$$\lim_{x\to 0}\left(1 + \frac{1}{x^2}\right) = \infty.$$

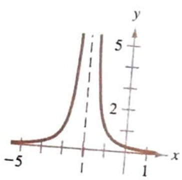
(b) Since $\lim_{x\to 1^-} (x^2 + 1) = 2$ and $\lim_{x\to 1^-} [1/(x-1)] = -\infty$, we can apply Property 2 of Theorem 2.13 to conclude that

$$\lim_{x\to 1^-} \frac{x^2+1}{1/(x-1)} = 0.$$

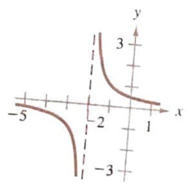
EXERCISES for Section 2.4

In Exercises 1 and 2, determine whether $f(x)$ approaches ∞ or $-\infty$ as x approaches -2 from the left and from the right.

1. $f(x) = \dfrac{1}{(x+2)^2}$



2. $f(x) = \dfrac{1}{x+2}$



In Exercises 3–6, determine whether $f(x)$ approaches ∞ or $-\infty$ as x approaches -3 from the left and from the right.

3. $f(x) = \dfrac{1}{x^2-9}$

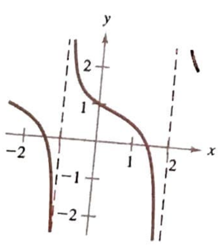
4. $f(x) = \dfrac{x}{x^2-9}$

5. $f(x) = \dfrac{x^3}{x^2-9}$

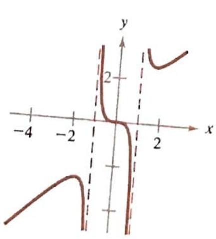
6. $f(x) = \dfrac{x^2}{x^2-9}$

In Exercises 7 and 8, find the vertical asymptotes of the given function.

7. $f(x) = \dfrac{x^2-2}{x^2-x-2}$



8. $f(x) = \dfrac{x^3}{x^2-1}$



In Exercises 9–18, find the vertical asymptotes (if any) of the given function.

9. $f(x) = \dfrac{1}{x^2}$

10. $f(x) = \dfrac{4}{(x-2)^3}$

11. $f(x) = \dfrac{x^2}{x^2+x-2}$

12. $f(x) = \dfrac{2+x}{1-x}$

13. $f(x) = \dfrac{x^3}{x^2-4}$

14. $f(x) = \dfrac{-4x}{x^2+4}$

15. $f(x) = 1 - \dfrac{4}{x^2}$

16. $f(x) = \dfrac{-2}{(x-2)^2}$

17. $f(x) = \dfrac{x}{x^2+x-2}$

18. $f(x) = \dfrac{1}{(x+3)^4}$

In Exercises 19–22, determine whether the given function has a vertical asymptote or a removable discontinuity at $x = -1$.

19. $f(x) = \dfrac{x^2-1}{x+1}$

20. $f(x) = \dfrac{x^2-6x-7}{x+1}$

21. $f(x) = \dfrac{x^2+1}{x+1}$

22. $f(x) = \dfrac{x-1}{x+1}$

In Exercises 23–32, find the indicated limit.

23. $\lim_{x\to 2^+} \dfrac{x-3}{x-2}$

24. $\lim_{x\to 1^+} \dfrac{2+x}{1-x}$

25. $\lim_{x\to 4^+} \dfrac{x^2}{x^2-16}$

26. $\lim_{x\to 4} \dfrac{x^2}{x^2+16}$

27. $\lim_{x\to 0^-}\left(1 + \dfrac{1}{x}\right)$

28. $\lim_{x\to 0^-}\left(x^2 - \dfrac{1}{x}\right)$

29. $\lim_{x\to 1} \dfrac{x^2-x}{(x^2+1)(x-1)}$

30. $\lim_{x\to 1} \dfrac{x^3-1}{x^2+x+1}$

31. $\lim_{x\to 1^+} \dfrac{x^2+x+1}{x^3-1}$

32. $\lim_{x\to 0^-} \dfrac{x^2-2x}{x^3}$

In Exercises 33–38, find the indicated limit (if it exists), given that

$$f(x) = \frac{1}{(x-4)^2} \quad \text{and} \quad g(x) = x^2 - 5x.$$

33. $\lim_{x\to 4} f(x)$

34. $\lim_{x\to 4} g(x)$

35. $\lim_{x\to 4} [f(x) + g(x)]$

36. $\lim_{x\to 4} [f(x)g(x)]$

37. $\lim_{x\to 4} \left[\frac{f(x)}{g(x)}\right]$

38. $\lim_{x\to 4} \left[\frac{g(x)}{f(x)}\right]$

39. The cost in dollars of removing p percent of the air pollutants from the stack emission of a utility company that burns coal to generate electricity is

$$C = \frac{80{,}000p}{100 - p}, \quad 0 \le p < 100.$$

(a) Find the cost of removing 15 percent.
(b) Find the cost of removing 50 percent.
(c) Find the cost of removing 90 percent.
(d) Find the limit of C as $x \to 100^-$.

40. The cost in millions of dollars for the federal government to seize x percent of a certain illegal drug as it enters the country is given by

$$C = \frac{528x}{100 - x}, \quad 0 \le x < 100.$$

(a) Find the cost of seizing 25 percent.
(b) Find the cost of seizing 50 percent.
(c) Find the cost of seizing 75 percent.
(d) Find the limit of C as $x \to 100^-$.

41. A 25-foot ladder is leaning against a house, as shown in the figure. If the base of the ladder is pulled away from the house at a rate of 2 feet per second, the top will move down the wall at a rate of

$$r = \frac{2x}{\sqrt{625 - x^2}} \text{ ft/sec.}$$

(a) Find the rate when x is 7 feet.
(b) Find the rate when x is 15 feet.
(c) Find the limit of r as $x \to 25^-$.

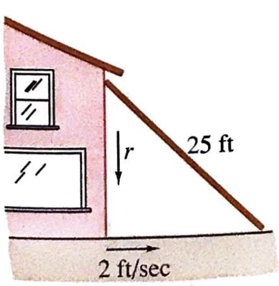


FIGURE FOR 41

42. **Coulomb's Law** states that the force F of a point charge q_1 on a point charge q_2, when the charges are r units apart, is proportional to the product of the charges and inversely proportional to the square of the distance between them. If a point particle with a charge of $+1$ is placed on a line between two particles 5 units apart, each with a charge of -1, the net force on the particle with a positive charge is given by

$$F = -\frac{k}{x^2} + \frac{k}{(x-5)^2}, \quad 0 < x < 5$$

where x is the distance shown in the figure. Sketch the graph of F.

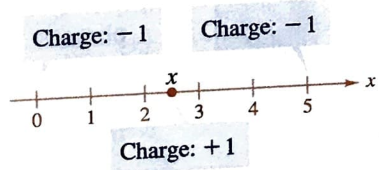


FIGURE FOR 42

43. Use a computer or graphics calculator to sketch the graph of the function

$$f(x) = \frac{1}{x^2 - 25}$$

and find $\lim_{x\to 5^-} f(x)$.

44. Find functions f and g such that

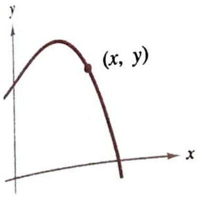
$$\lim_{x\to c} f(x) = \infty \quad \text{and} \quad \lim_{x\to c} g(x) = \infty$$

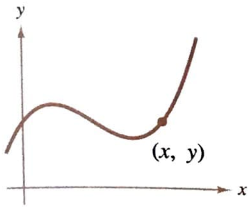
but

$$\lim_{x\to c} [f(x) - g(x)] \ne 0.$$

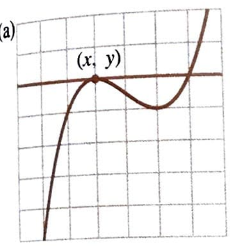
EXERCISES for Section 3.1

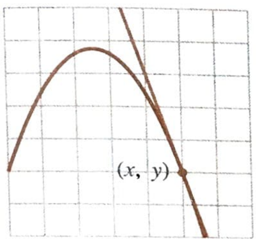
In Exercises 1 and 2, trace the curve on another piece of paper and sketch the tangent line at the point (x, y).

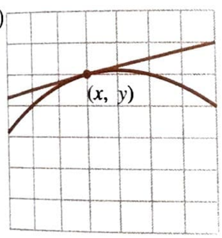
1.


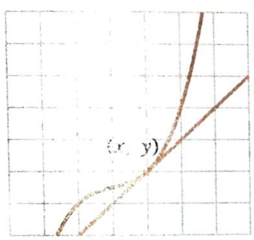
2.


In Exercises 3 and 4, estimate the slope of the curve at the point (x, y).

3. (a)


(b)


4. (a)


(b)


In Exercises 5–14, use the definition of the derivative to find $f'(x)$.

5. $f(x) = 3$
6. $f(x) = 3x + 2$
7. $f(x) = -5x$
8. $f(x) = 1 - x^2$
9. $f(x) = 2x^2 + x - 1$
10. $f(x) = \sqrt{x - 4}$
11. $f(x) = \dfrac{1}{x - 1}$
12. $f(x) = \dfrac{1}{x^2}$
13. $f(t) = t^3 - 12t$
14. $f(t) = t^3 + t^2$

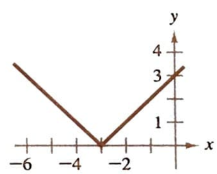
In Exercises 15–20, find the equation of the tangent line to the graph of f at the indicated point. Then verify your answer by sketching both the graph of f and the tangent line.

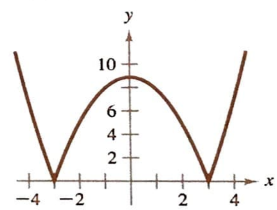
Function	Point of Tangency
15. $f(x) = x^2 + 1$	$(2, 5)$
16. $f(x) = x^2 + 2x + 1$	$(-3, 4)$
17. $f(x) = x^3$	$(2, 8)$
18. $f(x) = x^3$	$(-2, -8)$
19. $f(x) = \sqrt{x + 1}$	$(3, 2)$
20. $f(x) = \dfrac{1}{x + 1}$	$(0, 1)$

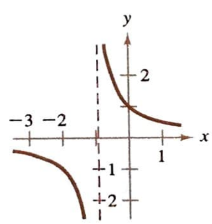
In Exercises 21–26, use the alternate form of the derivative (Theorem 3.1) to find the derivative at $x = c$ (if it exists).

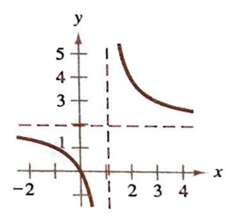
21. $f(x) = x^2 - 1,\ c = 2$
22. $f(x) = x^3 + 2x,\ c = 1$
23. $f(x) = x^3 + 2x^2 + 1,\ c = -2$
24. $f(x) = \dfrac{1}{x},\ c = 3$
25. $f(x) = (x - 1)^{2/3},\ c = 1$
26. $f(x) = |x - 2|,\ c = 2$

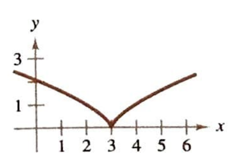
In Exercises 27–36, find every point at which the function is differentiable.

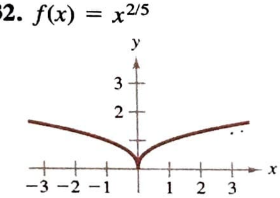
27. $f(x) = |x + 3|$


28. $f(x) = |x^2 - 9|$


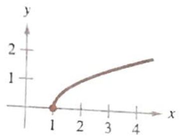
29. $f(x) = \dfrac{1}{x + 1}$


30. $f(x) = \dfrac{2x}{x - 1}$


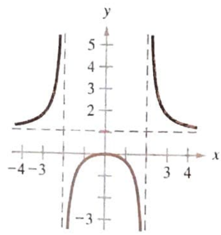
31. $f(x) = (x - 3)^{2/3}$


32. $f(x) = x^{2/5}$


33. $f(x) = \sqrt{x - 1}$



34. $f(x) = \dfrac{x^2}{x^2 - 4}$



35. $f(x) = \begin{cases} 4 - x^2, & 0 < x \\ x^2 - 4, & x \leq 0 \end{cases}$

36. $f(x) = \begin{cases} x^2 - 2x, & x > 1 \\ x^3 - 3x^2 + 3x, & x \leq 1 \end{cases}$

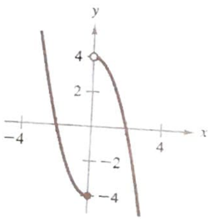


FIGURE FOR 35

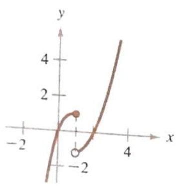


FIGURE FOR 36

37. Assume $f'(c) = 3$. Find $f'(-c)$ for the following conditions.

(a) f is an odd function.

(b) f is an even function.

38. Sketch the graph of f and f' on the same set of axes for each of the following.

(a) $f(x) = x^2$ (b) $f(x) = x^3$

In Exercises 39–42, find the derivatives from the left and from the right at $x = 1$ (if they exist). Is the function differentiable at $x = 1$?

39. $f(x) = \sqrt{1 - x^2}$

40. $f(x) = \begin{cases} x - 1, & x \leq 1 \\ (x - 1)^2, & x > 1 \end{cases}$

41. $f(x) = \begin{cases} (x - 1)^3, & x \leq 1 \\ (x - 1)^2, & x > 1 \end{cases}$

42. $f(x) = \begin{cases} x, & x \leq 1 \\ x^2, & x > 1 \end{cases}$

In Exercises 43 and 44, find an equation of the line that is tangent to the graph of f and parallel to the given line.

	Function	*Line*
43.	$f(x) = x^3$	$3x - y + 1 = 0$
44.	$f(x) = \dfrac{1}{\sqrt{x}}$	$x + 2y - 6 = 0$

In Exercises 45 and 46, find the equations of the two tangent lines to the graph of f that pass through the indicated point.

45. $f(x) = 4x - x^2$ **46.** $f(x) = x^2$

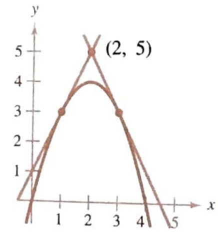


FIGURE FOR 45

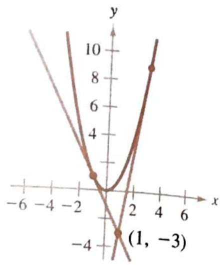


FIGURE FOR 46

In Exercises 47–50, determine whether the statement is true or false.

47. The slope of the graph of $y = x^2$ is different at every point on the curve.

48. If a function is continuous at a point, then it is differentiable at that point.

49. If a function is differentiable at a point, then it is continuous at that point.

50. If a function has derivatives from both the right and the left at a point, then it is differentiable at that point.

In Exercises 51 and 52, use a computer or graphics calculator to sketch the graph of f over the interval $[-2, 2]$ and complete the following table.

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
$f(x)$									
$f'(x)$									

51. $f(x) = \frac{1}{4}x^3$ **52.** $f(x) = \dfrac{4}{x}$

In Exercises 53 and 54, consider the functions $f(x)$ and $S_{\Delta x}(x)$ where

$$S_{\Delta x}(x) = \frac{f(2 + \Delta x) - f(2)}{\Delta x}(x - 2) + f(2).$$

(a) Use a graphing utility to graph f and $S_{\Delta x}$ on the same coordinate axes for $\Delta x = 1$, 0.5, and 0.1.

(b) Give a written description of the graphs of S for the different values of Δx.

53. $f(x) = 4 - (x - 3)^2$ **54.** $f(x) = x + \dfrac{1}{x}$

TABLE 3.4

t (min)	0	10	20	30	40	50	60	70	80	90	100	110	120
C (mg)	0	2	17	37	55	73	89	103	111	113	113	103	68

SOLUTION

For the interval [0, 10], the average rate of change is

$$\frac{\Delta C}{\Delta t} = \frac{2-0}{10-0} = \frac{2}{10} = 0.2 \text{ mg/min}.$$

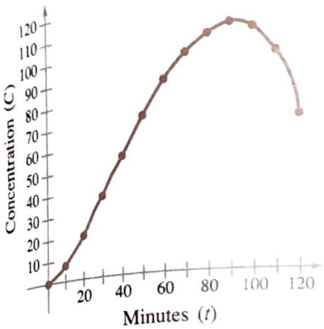
For the interval [0, 20], the average rate of change is

$$\frac{\Delta C}{\Delta t} = \frac{17-0}{20-0} = \frac{17}{20} = 0.85 \text{ mg/min}.$$

For the interval [100, 110], the average rate of change is

$$\frac{\Delta C}{\Delta t} = \frac{103-113}{110-100} = \frac{-10}{10} = -1 \text{ mg/min}.$$

Note in Figure 3.15 that the average rate of change is positive when the concentration increases and negative when the concentration decreases.



Drug Concentration in Bloodstream

FIGURE 3.15

To conclude this section, we give a summary concerning the derivative and its interpretations.

INTERPRETATIONS OF THE DERIVATIVE

If the function given by $y = f(x)$ is differentiable at x, then its derivative

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

denotes both

1. the *slope* of the graph of f at x and
2. the *instantaneous rate of change* in y with respect to x.

EXERCISES for Section 3.2

In Exercises 1–6, find the average rate of change of the given function over the indicated interval. Compare this average rate of change to the instantaneous rates of change at the endpoints of the interval.

Function	*Interval*
1. $f(t) = 2t + 7$	$[1, 2]$
2. $f(t) = 3t - 1$	$\left[0, \frac{1}{3}\right]$
3. $f(x) = \dfrac{1}{x+1}$	$[0, 3]$
4. $f(x) = \dfrac{-1}{x}$	$[1, 2]$
5. $f(t) = t^2 - 3$	$[2, 2.1]$
6. $f(x) = x^2 - 6x - 1$	$[-1, 3]$

7. The height s at time t of a silver dollar dropped from the World Trade Center is given by $s(t) = -16t^2 + 1350$, where s is measured in feet and t is measured in seconds $[s'(t) = -32t]$.
   (a) Find the average velocity on the interval $[1, 2]$.
   (b) Find the instantaneous velocity when $t = 1$ and $t = 2$.
   (c) How long will it take the dollar to hit the ground?
   (d) Find the velocity of the dollar when it hits the ground.

8. An automobile's velocity starting from rest is given by

$$v = \frac{100t}{2t + 15}$$

where v is measured in feet per second. Find the acceleration at the following times.
   (a) 5 seconds (b) 10 seconds (c) 20 seconds

In Exercises 9–14, use the following position and velocity functions for free-falling objects.

$$s(t) = -16t^2 + v_0 t + s_0$$
$$s'(t) = -32t + v_0$$

9. A projectile is shot upward from the surface of the earth with an initial velocity of 384 feet per second. What is its velocity after 5 seconds? After 10 seconds?

10. Repeat Exercise 9 for an initial velocity of 256 feet per second.

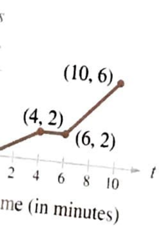
11. A pebble is dropped from a height of 600 feet. Find the pebble's velocity when it hits the ground.

2. A ball is thrown straight down from the top of a 220-foot building with an initial velocity of -22 feet per second. What is its velocity after 3 seconds? What is its velocity after falling 108 feet?

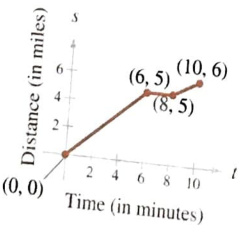
. To estimate the height of a building, a stone is dropped from the top of the building into a pool of water at ground level. How high is the building if the splash is seen 6.8 seconds after the stone is dropped?

A ball is dropped from a height of 100 feet. One second later another ball is dropped from a height of 75 feet. Which ball hits the ground first?

xercises 15 and 16, the graphs of position functions given. They represent the distance in miles that a on drives during a 10-minute trip to work. Make ches of the corresponding velocity functions.

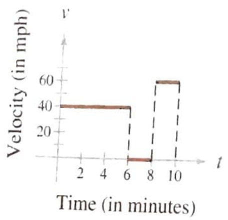


16.

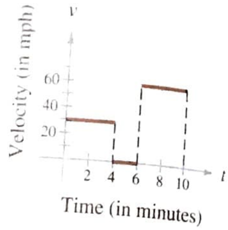


In Exercises 17 and 18, the graphs of velocity functions are given. They represent the velocity in miles per hour during a 10-minute drive to work. Make sketches of the corresponding position functions.

17.



18.



In Exercises 19–24, find the indicated derivative.

Given	Find
19. $f'(x) = x^2$	$f''(x)$
20. $f''(x) = x^3$	$f'''(x)$
21. $f''(x) = 2 - \frac{2}{x}$	$f'''(x)$
22. $f'''(x) = 2\sqrt{x - 1}$	$f^{(4)}(x)$
23. $f^{(4)}(x) = 2x + 1$	$f^{(6)}(x)$
24. $f(x) = 2x^2 - 2$	$f''(x)$

25. The annual inventory cost for a certain manufacturer is given by

$$C = \frac{1{,}008{,}000}{Q} + 6.3Q$$

where Q is the order size when the inventory is replenished. Find the change in annual cost when Q is increased from 350 to 351 and compare this with the rate of change

$$\frac{dC}{dQ} = -\frac{1{,}008{,}000}{Q^2} + 6.3$$

when $Q = 350$.

26. A car is driven 15,000 miles a year and gets x miles per gallon. Assume that the average fuel cost is \$1.10 per gallon. Find the annual cost of fuel C as a function of x and use this function to complete the following table.

x	10	15	20	25	30	35	40
C							
$\frac{dC}{dx}$							

27. When a guitar string is plucked, it vibrates with a frequency of

$$F = 200\sqrt{T}$$

where F is measured in vibrations per second and the

(b) The velocity at time t is given by the derivative

$$s'(t) = -32t + 16.$$

Therefore, the velocity at time $t = 2$ is

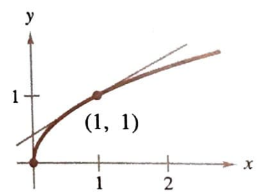
$$s'(2) = -32(2) + 16 = -48 \text{ ft/sec.}$$

REMARK In Figure 3.18, note that the diver moves upward for the first half-second. This corresponds to the fact that the velocity is positive for $0 < t < \frac{1}{2}$.

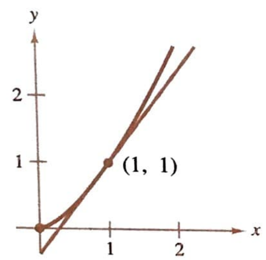
EXERCISES for Section 3.3

In Exercises 1 and 2, find the slope of the tangent line to $y = x^n$ at the point $(1, 1)$.

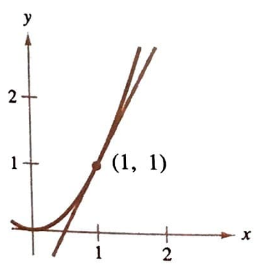
1. (a) $y = x^{1/2}$



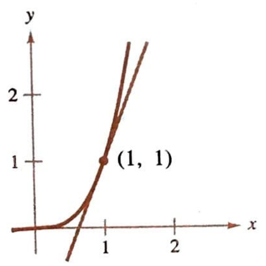
(b) $y = x^{3/2}$



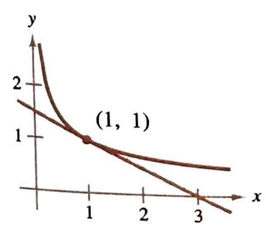
(c) $y = x^2$



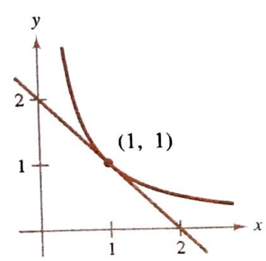
(d) $y = x^3$



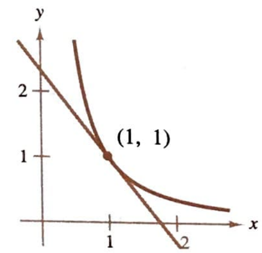
2. (a) $y = x^{-1/2}$



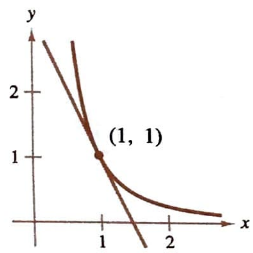
(b) $y = x^{-1}$



(c) $y = x^{-3/2}$



(d) $y = x^{-2}$



In Exercises 3–12, find the derivative of the given function.

3. $y = 3$

4. $f(x) = -2$

5. $f(x) = x + 1$

6. $g(x) = 3x - 1$

7. $g(x) = x^2 + 4$

8. $y = t^2 + 2t - 3$

9. $f(t) = -2t^2 + 3t - 6$

10. $y = x^3 - 9$

11. $s(t) = t^3 - 2t + 4$

12. $f(x) = 2x^3 - x^2 + 3x$

In Exercises 13–18, find the value of the derivative of the given function at the indicated point.

	Function	*Point*
13.	$f(x) = \dfrac{1}{x}$	$(1, 1)$
14.	$f(t) = 3 - \dfrac{3}{5t}$	$\left(\dfrac{3}{5}, 2\right)$
15.	$f(x) = -\dfrac{1}{2} + \dfrac{7}{5}x^3$	$\left(0, -\dfrac{1}{2}\right)$
16.	$y = 3x\left(x^2 - \dfrac{2}{x}\right)$	$(2, 18)$
17.	$y = (2x + 1)^2$	$(0, 1)$
18.	$f(x) = 3(5 - x)^2$	$(5, 0)$

In Exercises 19–30, find $f'(x)$.

19. $f(x) = x^2 - \dfrac{4}{x}$
20. $f(x) = x^2 - 3x - 3x^{-2}$
21. $f(x) = x^3 - 3x - \dfrac{2}{x^4}$
22. $f(x) = \dfrac{2x^2 - 3x + 1}{x}$
23. $f(x) = \dfrac{x^3 - 3x^2 + 4}{x^2}$
24. $f(x) = (x^2 + 2x)(x + 1)$
25. $f(x) = x(x^2 + 1)$
26. $f(x) = x + \dfrac{1}{x^2}$
27. $f(x) = x^{4/5}$
28. $f(x) = x^{1/3} - 1$
29. $f(x) = \sqrt[3]{x} + \sqrt[5]{x}$
30. $f(x) = \dfrac{1}{\sqrt[3]{x^2}}$

In Exercises 31–36, complete the table, using Example 6 as a model.

Function	*Rewrite*	*Derivative*	*Simplify*
31. $y = \dfrac{1}{3x^3}$			
32. $y = \dfrac{2}{3x^2}$			
33. $y = \dfrac{1}{(3x)^3}$			
34. $y = \dfrac{\pi}{(3x)^2}$			
35. $y = \dfrac{\sqrt{x}}{x}$			
36. $y = \dfrac{4}{x^{-3}}$			

n Exercises 37 and 38, find an equation of the tangent ine to the given function at the indicated point.

7. $y = x^4 - 3x^2 + 2,\ (1, 0)$
8. $y = x^3 + x,\ (-1, -2)$

n Exercises 39–42, determine the point(s) (if any) at hich the given function has a horizontal tangent line.

. $y = x^4 - 3x^2 + 2$
40. $y = x^3 + x$
. $y = \dfrac{1}{x^2}$
42. $y = x^2 + 1$

43. Sketch the graphs of the two equations $y = x^2$ and $y = -x^2 + 6x - 5$, and sketch the two lines that are tangent to both graphs. Find the equations of these lines.

44. Show that the graphs of the two equations $y = x$ and $y = 1/x$ have tangent lines that are perpendicular to each other at their points of intersection.

45. The area of a square with sides of length s is given by $A = s^2$. Find the rate of change of the area with respect to s when $s = 4$.

46. The volume of a cube with sides of length s is given by $V = s^3$. Find the rate of change of the volume with respect to s when $s = 4$.

47. A company finds that charging p dollars per unit produces a monthly revenue

$$R = 12{,}000p - 1{,}000p^2, \quad 0 \le p \le 12.$$

(Note that the revenue is zero when $p = 12$ since no one is willing to pay that much.) Find the rate of change of R with respect to p when p has the following values.
(a) $p = 1$ (b) $p = 4$
(c) $p = 6$ (d) $p = 10$

48. Suppose that the profit P obtained in selling x units of a certain item each week is given by

$$P = 50\sqrt{x} - 0.5x - 500, \quad 0 \le x \le 8000.$$

Find the rate of change of P with respect to x when
(a) $x = 900$ (b) $x = 1600$
(c) $x = 2500$ (d) $x = 3600$
(Note that the eventual decline in profit occurs because the only way to sell larger quantities is to decrease the price per unit.)

49. Suppose that the effectiveness E of a painkilling drug t hours after entering the bloodstream is given by

$$E = \frac{1}{27}(9t + 3t^2 - t^3), \quad 0 \le t \le 4.5.$$

Find the rate of change of E with respect to t when
(a) $t = 1$ (b) $t = 2$
(c) $t = 3$ (d) $t = 4$

50. In a certain chemical reaction, the amount in grams Q of a substance produced in t hours is given by the equation

$$Q = 16t - 4t^2, \quad 0 < t \le 2.$$

Find the rate in grams per hour at which the substance is being produced when t has the following values.
(a) $t = \dfrac{1}{2}$ (b) $t = 1$ (c) $t = 2$

Differentiate

$$y = \left(\frac{3x - 1}{x^2 + 3}\right)^2.$$

SOLUTION

$$\frac{dy}{dx} = \overbrace{2}^{n}\overbrace{\left(\frac{3x - 1}{x^2 + 3}\right)}^{u^{n-1}}\overbrace{\frac{d}{dx}\left[\frac{3x - 1}{x^2 + 3}\right]}^{u'}$$

$$= \left[\frac{2(3x - 1)}{x^2 + 3}\right]\left[\frac{(x^2 + 3)(3) - (3x - 1)(2x)}{(x^2 + 3)^2}\right]$$

$$= \frac{2(3x - 1)(3x^2 + 9 - 6x^2 + 2x)}{(x^2 + 3)^3}$$

$$= \frac{2(3x - 1)(-3x^2 + 2x + 9)}{(x^2 + 3)^3}$$

Try finding y' using the Quotient Rule on $y = (3x - 1)^2/(x^2 + 3)^2$ and compare the results.

EXERCISES for Section 3.5

In Exercises 1–6, complete the table using Example 2 as a model.

$y = f(g(x))$	$u = g(x)$	$y = f(u)$
1. $y = (6x - 5)^4$		
2. $y = \dfrac{1}{\sqrt{x + 1}}$		
3. $y = \sqrt{x^2 - 1}$		
4. $y = \left(\dfrac{3x}{2}\right)^2$		
5. $y = (x^2 - 3x + 4)^6$		
6. $y = (5x - 2)^{3/2}$		

In Exercises 7–44, find the derivative.

7. $y = (2x - 7)^3$

8. $y = (3x^2 + 1)^4$

9. $g(x) = 3(9x - 4)^4$

10. $f(x) = 2(x^2 - 1)^3$

11. $y = \dfrac{1}{x - 2}$

12. $s(t) = \dfrac{1}{t^2 + 3t - 1}$

13. $f(t) = \left(\dfrac{1}{t - 3}\right)^2$

14. $y = -\dfrac{4}{(t + 2)^2}$

15. $f(x) = \dfrac{3}{x^3 - 4}$

16. $f(x) = \dfrac{1}{(x^2 - 3x)^2}$

17. $f(x) = x^2(x - 2)^4$

18. $f(x) = x(3x - 9)^3$

19. $f(t) = \sqrt{1 - t}$

20. $g(x) = \sqrt{3 - 2x}$

21. $s(t) = \sqrt{t^2 + 2t - 1}$ **22.** $y = \sqrt[3]{3x^3 + 4x}$

23. $y = \sqrt[3]{9x^2 + 4}$ **24.** $g(x) = \sqrt{x^2 - 2x + 1}$

25. $y = 2\sqrt{4 - x^2}$ **26.** $f(x) = -3\sqrt[4]{2 - 9x}$

27. $f(x) = (9 - x^2)^{2/3}$ **28.** $f(t) = (9t + 2)^{2/3}$

29. $y = \dfrac{1}{\sqrt{x + 2}}$ **30.** $g(t) = \sqrt{\dfrac{1}{t^2 - 2}}$

31. $y = x\sqrt{1 - x^2}$ **32.** $y = x^2\sqrt{9 - x^2}$

33. $y = \dfrac{x}{\sqrt{x^2 + 1}}$ **34.** $y = \dfrac{x^2}{\sqrt{x^2 + 9}}$

35. $f(t) = \dfrac{3t + 2}{t - 1}$ **36.** $f(x) = \dfrac{x + 1}{2x - 3}$

In Exercises 37–44, use a symbolic differentiation utility to find the first derviative of the function. Then use the utility to graph the function and its derivative on the same set of coordinate axes. Describe the behavior of the function that corresponds to any zeros of the graph of the derivative.

37. $y = \dfrac{\sqrt{x} + 1}{x^2 + 1}$ **38.** $y = \sqrt{\dfrac{2x}{x + 1}}$

39. $g(t) = \dfrac{3t^2}{\sqrt{t^2 + 2t - 1}}$ **40.** $f(x) = \sqrt{x}(2 - x)^2$

41. $y = \sqrt{\dfrac{x + 1}{x}}$ **42.** $y = (t^2 - 9)\sqrt{t + 2}$

43. $s(t) = \dfrac{-2(2 - t)\sqrt{1 + t}}{3}$

44. $g(x) = \sqrt{x - 1} + \sqrt{x + 1}$

In Exercises 45 and 46, find an equation of the tangent line to the graph of f at the given point.

Function	*Point*
45. $f(x) = \sqrt{3x^2 - 2}$	$(3, 5)$
46. $f(x) = x\sqrt{x^2 + 5}$	$(2, 6)$

In Exercises 47–50, find the second derivative of the given function.

47. $f(x) = 2(x^2 - 1)^3$ **48.** $f(x) = \dfrac{1}{x - 2}$

49. $f(x) = \sqrt{x^2 + x + 1}$ **50.** $f(t) = \dfrac{\sqrt{t^2 + 1}}{t}$

51. Let u be a differentiable function of x. Use the fact that $|u| = \sqrt{u^2}$ to prove that

$$\frac{d}{dx}[|u|] = u'\frac{u}{|u|}, \quad u \neq 0.$$

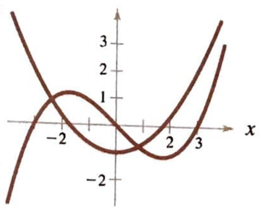
In Exercises 52–54, use the result of Exercise 51 to find the derivative of the given function.

52. $f(x) = |x^2 - 4|$

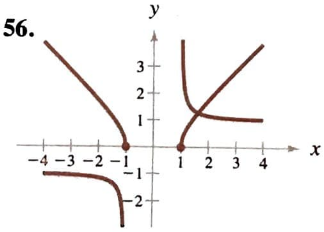
53. $f(x) = |x^3 + x|$ **54.** $f(x) = \left|\dfrac{4}{x}\right|$

In Exercises 55 and 56, the graphs of a function f and its derivative f' are given. Label the graphs as f or f' and write a short paragraph stating the criteria used in making the selection.

55.



56.



57. *(Doppler effect)* The frequency F of a fire truck siren heard by a stationary observer is given by

$$F = \frac{132{,}400}{331 \pm v}$$

where $\pm v$ represents the velocity of the accelerating fire truck (see figure). Find the rate of change of F with respect to v when (a) the fire truck is approaching at a velocity of 30 m/s [use $-v$], and then when (b) the fire truck is moving away at a velocity of 30 m/s [use $+v$].

Doppler Effect

3.6 Implicit Differentiation

Implicit and explicit functions ▪ Implicit differentiation

So far, our equations involving two variables were generally expressed in the **explicit form** $y = f(x)$. That is, one of the two variables was explicitly given in terms of the other. For example,

$$y = 3x - 5, \qquad s = -16t^2 + 20t, \qquad u = 3w - w^2$$

all are written in explicit form, and we say that y, s, and u are functions of x, t, and w, respectively.

A television camera at ground level is filming the lift-off of a space shuttle that is rising vertically according to the position equation $s = 50t^2$, where s is measured in feet and t is measured in seconds. The camera is 2000 feet from the launch pad. Find the rate of change in the distance between the camera and the base of the shuttle 10 seconds after lift-off. (Assume that the camera and the base of the shuttle are level with each other when $t = 0$.)

SOLUTION

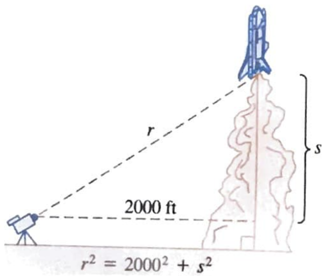


FIGURE 3.31

1. We let r be the distance between the camera and the base of the shuttle, as shown in Figure 3.31. Then we can find the velocity of the rocket by differentiating s with respect to t to obtain $ds/dt = 100t$. Thus, we have the following model.

   Given: $\dfrac{ds}{dt} = 100t = \text{velocity}$

   Find: $\dfrac{dr}{dt}$ when $t = 10$

2. Using Figure 3.31 we relate s and r by the equation

   $$r^2 = 2000^2 + s^2.$$

3. Implicit differentiation with respect to t yields

   $$2r\frac{dr}{dt} = 2s\frac{ds}{dt}$$

   $$\frac{dr}{dt} = \frac{s}{r}\cdot\frac{ds}{dt} = \frac{s}{r}(10t).$$

4. Now, when $t = 10$, we know that $s = 50(10^2) = 5000$, and we have

   $$r = \sqrt{2000^2 + 5000^2} = 1000\sqrt{29}.$$

   Finally, the rate of change of r when $t = 10$ is

   $$\frac{dr}{dt} = \frac{5000}{1000\sqrt{29}}(100)(10) = 928.48 \text{ ft/sec.}$$

EXERCISES for Section 3.7

In Exercises 1–4, assume that x and y are both differentiable functions of t and find the indicated values of dy/dt and dx/dt.

Equation	Find	Given
1. $y = \sqrt{x}$	(a) $\dfrac{dy}{dt}$ when $x = 4$	$\dfrac{dx}{dt} = 3$
	(b) $\dfrac{dx}{dt}$ when $x = 25$	$\dfrac{dy}{dt} = 2$

Equation	Find	Given
2. $y = x^2 - 3x$	(a) $\dfrac{dy}{dt}$ when $x = 3$	$\dfrac{dx}{dt} = 2$
	(b) $\dfrac{dx}{dt}$ when $x = 1$	$\dfrac{dy}{dt} = 5$
3. $xy = 4$	(a) $\dfrac{dy}{dt}$ when $x = 8$	$\dfrac{dx}{dt} = 10$
	(b) $\dfrac{dx}{dt}$ when $x = 1$	$\dfrac{dy}{dt} = -6$

	Equation	Find	Given
4.	$x^2 + y^2 = 25$	(a) $\dfrac{dy}{dt}$ when $x = 3$, $y = 4$	$\dfrac{dx}{dt} = 8$
		(b) $\dfrac{dx}{dt}$ when $x = 4$, $y = 3$	$\dfrac{dy}{dt} = -2$

5. The radius r of a circle is increasing at a rate of 2 inches per minute. Find the rate of change of the area when (a) $r = 6$ inches and (b) $r = 24$ inches.

6. The radius r of a sphere is increasing at a rate of 2 inches per minute. Find the rate of change of the volume when (a) $r = 6$ inches and (b) $r = 24$ inches.

7. Let A be the area of a circle of radius r that is changing with respect to time. If dr/dt is constant, is dA/dt constant? Explain why or why not.

8. Let V be the volume of a sphere of radius r that is changing with respect to time. If dr/dt is constant, is dV/dt constant? Explain why or why not.

9. A spherical balloon is inflated with gas at the rate of 20 cubic feet per minute. How fast is the radius of the balloon increasing at the instant the radius is (a) 1 foot and (b) 2 feet?

10. The formula for the volume of a cone is

$$V = \frac{1}{3}\pi r^2 h.$$

Find the rate of change of the volume if dr/dt is 2 inches per minute and $h = 3r$ when (a) $r = 6$ inches and (b) $r = 24$ inches.

11. At a sand and gravel plant, sand is falling off a conveyor and onto a conical pile at the rate of 10 cubic feet per minute. The diameter of the base of the cone is approximately three times the altitude. At what rate is the height of the pile changing when it is 15 feet high?

12. A conical tank (with vertex down) is 10 feet across the top and 12 feet deep. If water is flowing into the tank at the rate of 10 cubic feet per minute, find the rate of change of the depth of the water the instant it is 8 feet deep.

13. All edges of a cube are expanding at the rate of 3 centimeters per second. How fast is the volume changing when each edge is (a) 1 centimeter and (b) 10 centimeters?

14. The conditions are the same as in Exercise 13. Now measure how fast the *surface* area is changing when each edge is (a) 1 centimeter and (b) 10 centimeters.

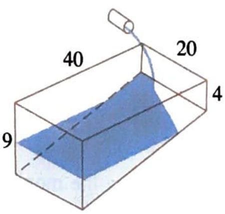
15. A point is moving along the graph of $y = x^2$ so that dx/dt is 2 centimeters per minute. Find dy/dt when (a) $x = 0$ and (b) $x = 3$.

16. The conditions are the same as in Exercise 15, but now measure the rate of change of the distance between the point and the origin.

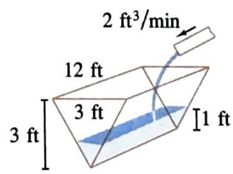
17. A point is moving along the graph of $y = 1/(1 + x^2)$ so that $dx/dt = 2$ centimeters per minute. Find dy/dt for the following values of x.
(a) $x = -2$
(b) $x = 0$
(c) $x = 2$
(d) $x = 10$

18. A point is moving along the graph of $y = x^3$ so that $dx/dt = 2$ centimeters per minute. Find dy/dt for the following values of x.
(a) $x = -2$
(b) $x = 1$
(c) $x = 0$
(d) $x = 3$

19. A swimming pool is 40 feet long, 20 feet wide, 4 feet deep at the shallow end, and 9 feet deep at the deep end (see figure). Water is being pumped into the pool at 10 cubic feet per minute, and there is 4 feet of water at the deep end.
(a) What percentage of the pool is filled?
(b) At what rate is the water level rising?



20. A trough is 12 feet long and 3 feet across the top (see figure). Its ends are isosceles triangles with an altitude of 3 feet. If water is being pumped into the trough at 2 cubic feet per minute, how fast is the water level rising when it is 1 foot deep?



21. A ladder 25 feet long is leaning against the wall of a house (see figure). The base of the ladder is pulled away from the wall at a rate of 2 feet per second. How fast is the top moving down the wall when the base of the ladder is (a) 7 feet, (b) 15 feet, and (c) 24 feet from the wall?