Chapter 11 Further Topics in Algebra

11.1 Sequences and Series

11.1 Practice Problems

1. $a_n = -2^n \Rightarrow a_1 = -2^1 = -2;\ a_2 = -2^2 = -4;$
$a_3 = -2^3 = -8;\ a_4 = -2^4 = -16$

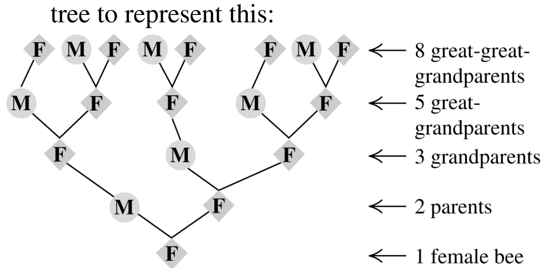
2.

n:	1	2	3	4	5	…	n
term:	0	$-\frac{1}{2}$	$\frac{2}{3}$	$-\frac{3}{4}$	$\frac{4}{5}$	…	a_n

Since the terms alternate, and the negative terms are the even-numbered terms, use the factor $(-1)^{n+1}$ to alternate the signs. The absolute value of each term is $\frac{n-1}{n} = 1 - \frac{1}{n}$, so the general term is $(-1)^{n+1}\left(1 - \frac{1}{n}\right)$.

3. $a_1 = -3,\ a_{n+1} = 2a_n + 5$
$a_2 = 2a_1 + 5 = 2(-3) + 5 = -1$
$a_3 = 2a_2 + 5 = 2(-1) + 5 = 3$
$a_4 = 2a_3 + 5 = 2(3) + 5 = 11$
$a_5 = 2a_4 + 5 = 2(11) + 5 = 27$

4. The female bee is a_0. Since female bees have two parents, $a_1 = 2$. The female parent has two parents and the male parent has one, so $a_2 = 3$. There are two females and one male in this generation, so the next generation has three females and two males. We can use a tree to represent this:



$a_k = a_{k-2} + a_{k-1}$ for $k \geq 3$.

5. a. $\frac{13!}{12!} = \frac{13 \cdot 12!}{12!} = 13$

b. $\frac{n!}{(n-3)!} = \frac{n(n-1)(n-2)(n-3)!}{(n-3)!} = n(n-1)(n-2)$

6. $a_n = \frac{(-1)^n 2^n}{n!} \Rightarrow a_1 = \frac{(-1)^1 2^1}{1!} = -2$
$a_2 = \frac{(-1)^2 2^2}{2!} = \frac{4}{2 \cdot 1} = 2$
$a_3 = \frac{(-1)^3 2^3}{3!} = \frac{-8}{3 \cdot 2 \cdot 1} = -\frac{8}{6} = -\frac{4}{3}$
$a_4 = \frac{(-1)^4 2^4}{4!} = \frac{16}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{16}{24} = \frac{2}{3}$
$a_5 = \frac{(-1)^5 2^5}{5!} = \frac{-32}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = -\frac{32}{120} = -\frac{4}{15}$

7. $\sum_{k=0}^{3} (-1)^k k!$
$= (-1)^0(0!) + (-1)^1(1!) + (-1)^2(2!) + (-1)^3(3!)$
$= 1 + (-1) + (2 \cdot 1) + (-1)(3 \cdot 2 \cdot 1) = -4$

8. $2 - 4 + 6 - 8 + 10 - 12 + 14$ alternates addition and subtraction of consecutive even integers from 2 to 14. The even integers can be represented as $2k$ with k ranging from 1 to 7. We alternate the signs using the factor $(-1)^{k+1}$. Thus, the expression is $\sum_{k=1}^{7} (-1)^{k+1}(2k)$.

11.1 Basic Concepts and Skills

1. An infinite sequence is a function whose domain is the set of positive integers.

2. If a sequence is defined by $a_n = 2n - \frac{10}{n}$, then $a_5 = 2(5) - \frac{10}{5} = 8$.

3. By definition, $0! = 1$.

4. $\sum_{k=0}^{5} \frac{2k+1}{2} = \frac{2(0)+1}{2} + \frac{2(1)+1}{2} + \frac{2(2)+1}{2} + \frac{2(3)+1}{2} + \frac{2(4)+1}{2} + \frac{2(5)+1}{2}$
$= \frac{1}{2} + \frac{3}{2} + \frac{5}{2} + \frac{7}{2} + \frac{9}{2} + \frac{11}{2} = 18$

5. False. $a_1 = 3,\ a_2 = a_1^2 = 3^2 = 9,$
$a_3 = a_2^2 = 9^2 = 81$

6. False.
$$\frac{(2n)!}{n!}=\frac{2n(2n-1)(2n-2)(2n-3)!}{n(n-1)(n-2)(n-3)!}\neq 2$$

7. $a_1=3(1)-2=1, a_2=3(2)-2=4,$
$a_3=3(3)-2=7, a_4=3(4)-2=10$

8. $a_1=2(1)+1=3, a_2=2(2)+1=5,$
$a_3=2(3)+1=7, a_4=2(4)+1=9$

9. $a_1=1-\frac{1}{1}=0, a_2=1-\frac{1}{2}=\frac{1}{2},$
$a_3=1-\frac{1}{3}=\frac{2}{3}, a_4=1-\frac{1}{4}=\frac{3}{4}$

10. $a_1=1+\frac{1}{1}=2, a_2=1+\frac{1}{2}=\frac{3}{2},$
$a_3=1+\frac{1}{3}=\frac{4}{3}, a_4=1+\frac{1}{4}=\frac{5}{4}$

11. $a_1=-1^2=-1, a_2=-2^2=-4,$
$a_3=-3^2=-9, a_4=-4^2=-16$

12. $a_1=1^3=1, a_2=2^3=8,$
$a_3=3^3=27, a_4=4^3=64$

13. $a_1=\frac{2(1)}{1+1}=1, a_2=\frac{2(2)}{2+1}=\frac{4}{3}, a_3=\frac{2(3)}{3+1}=\frac{3}{2},$
$a_4=\frac{2(4)}{4+1}=\frac{8}{5}$

14. $a_1=\frac{3(1)}{1^2+1}\Rightarrow a_1=\frac{3}{2}, a_2=\frac{3(2)}{2^2+1}=\frac{6}{5},$
$a_3=\frac{3(3)}{3^2+1}=\frac{9}{10}, a_4=\frac{3(4)}{4^2+1}=\frac{12}{17}$

15. $a_1=(-1)^{1+1}=1, a_2=(-1)^{2+1}=-1,$
$a_3=(-1)^{3+1}=1, a_4=(-1)^{4+1}=-1$

16. $a_1=(-3)^{1-1}=1, a_2=(-3)^{2-1}=-3,$
$a_3=(-3)^{3-1}=9, a_4=(-3)^{4-1}=-27$

17. $a_1=3-\frac{1}{2^1}=\frac{5}{2}, a_2=3-\frac{1}{2^2}=\frac{11}{4},$
$a_3=3-\frac{1}{2^3}=\frac{23}{8}, a_4=3-\frac{1}{2^4}=\frac{47}{16}$

18. $a_1=\left(\frac{3}{2}\right)^1=\frac{3}{2}, a_2=\left(\frac{3}{2}\right)^2=\frac{9}{4},$
$a_3=\left(\frac{3}{2}\right)^3=\frac{27}{8}, a_4=\left(\frac{3}{2}\right)^4=\frac{81}{16}$

19. $a_1=a_2=a_3=a_4=0.6$

20. $a_1=a_2=a_3=a_4=-0.4$

21. $a_1=\frac{(-1)^1}{1!}=-1, a_2=\frac{(-1)^2}{2!}=\frac{1}{2},$
$a_3=\frac{(-1)^3}{3!}=-\frac{1}{6}, a_4=\frac{(-1)^4}{4!}=\frac{1}{24}$

22. $a_1=\frac{1}{1!}=1, a_2=\frac{2}{2!}=1, a_3=\frac{3}{3!}=\frac{1}{2}, a_4=\frac{4}{4!}=\frac{1}{6}$

23. $a_1=(-1)^1 3^{-1}=-\frac{1}{3}, a_2=(-1)^2 3^{-2}=\frac{1}{9},$
$a_3=(-1)^3 3^{-3}=-\frac{1}{27}, a_4=(-1)^4 3^{-4}=\frac{1}{81}$

24. $a_1=(-1)^1 3^{1-1}=-1, a_2=(-1)^2 3^{2-1}=3,$
$a_3=(-1)^3 3^{3-1}=-9, a_4=(-1)^4 3^{4-1}=27$

25. $a_1=\frac{e^1}{2(1)}=\frac{e}{2}, a_2=\frac{e^2}{2(2)}=\frac{e^2}{4}, a_3=\frac{e^3}{2(3)}=\frac{e^3}{6},$
$a_4=\frac{e^4}{2(4)}=\frac{e^4}{8}$

26. $a_1=\frac{2^1}{e^1}=\frac{2}{e}, a_2=\frac{2^2}{e^2}=\frac{4}{e^2}, a_3=\frac{2^3}{e^3}=\frac{8}{e^3},$
$a_4=\frac{2^4}{e^4}=\frac{16}{e^4}$

27. $a_n=3n-2$ **28.** $a_n=2n+5$

29. $a_n=\frac{1}{n+1}$ **30.** $a_n=\frac{n+1}{n+2}$

31. $a_n=(-1)^{n+1}(2)$ **32.** $a_n=(-1)^n(3n)$

33. $a_n=\frac{3^{n-1}}{2^n}$ **34.** $a_n=\left(-\frac{1}{2}\right)^n$

35. $a_n=n(n+1)$ **36.** $a_n=\frac{(-1)^n}{n(n+1)}$

37. $a_n=2-\frac{(-1)^n}{n+1}$ **38.** $a_n=1+\frac{1}{n+1}$

39. $a_n=\frac{3^{n+1}}{n+1}$ **40.** $a_n=\left(\frac{e}{2}\right)^n$

41. $a_1=2, a_2=2+3=5, a_3=5+3=8,$
$a_4=8+3=11, a_5=11+3=14$

42. $a_1=5, a_2=5-1=4, a_3=4-1=3,$
$a_4=3-1=2, a_5=2-1=1$

43. $a_1 = 3, a_2 = 2(3) = 6, a_3 = 2(6) = 12,$
$a_4 = 2(12) = 24, a_5 = 2(24) = 48$

44. $a_1 = 1, a_2 = \frac{1}{2}(1) = \frac{1}{2}, a_3 = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4},$
$a_4 = \frac{1}{2}\left(\frac{1}{4}\right) = \frac{1}{8}, a_5 = \frac{1}{2}\left(\frac{1}{8}\right) = \frac{1}{16}$

45. $a_1 = 7, a_2 = -2(7) + 3 = -11,$
$a_3 = -2(-11) + 3 = 25, a_4 = -2(25) + 3 = -47,$
$a_5 = -2(-47) + 3 = 97$

46. $a_1 = -4, a_2 = -3(-4) - 5 = 7,$
$a_3 = -3(7) - 5 = -26, a_4 = -3(-26) - 5 = 73,$
$a_5 = -3(73) - 5 = -224$

47. $a_1 = 2, a_2 = \frac{1}{2}, a_3 = \frac{1}{1/2} = 2, a_4 = \frac{1}{2},$
$a_5 = \frac{1}{1/2} = 2$

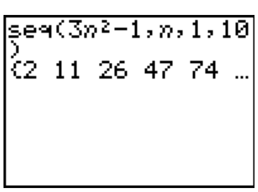
48. $a_1 = -1, a_2 = \frac{-1}{-1} = 1, a_3 = \frac{-1}{1} = -1, a_4 = \frac{-1}{-1} = 1,$
$a_5 = \frac{-1}{1} = -1$

49. $a_1 = 25, a_2 = \frac{(-1)^1}{5(25)} = -\frac{1}{125},$
$a_3 = \frac{(-1)^2}{5(-1/125)} = -25, a_4 = \frac{(-1)^3}{5(-25)} = \frac{1}{125},$
$a_5 = \frac{(-1)^4}{5(1/125)} = 25$

50. $a_1 = 12, a_2 = \frac{(-1)^1}{3(12)} = -\frac{1}{36},$
$a_3 = \frac{(-1)^2}{3(-1/36)} = -12, a_4 = \frac{(-1)^3}{3(-12)} = \frac{1}{36},$
$a_5 = \frac{(-1)^4}{3(1/36)} = 12$

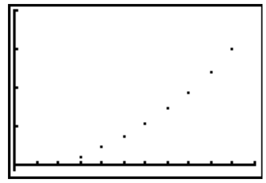
In exercises 51–56, we show only the first screen to determine the terms of the sequence before we list the ten terms.

51. a.

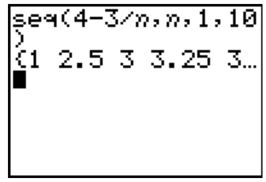


2, 11, 26, 47, 74, 107, 146, 191, 242, 299

b.

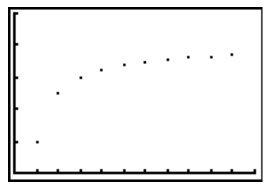


52. a.

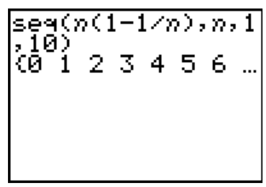


1, 2.5, 3, 3.25, 3.4, 3.5, 3.571, 3.625, 3.667, 3.7 (rounded to three decimal places)

b.

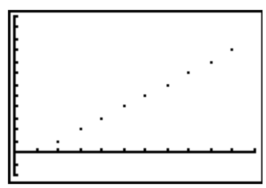


53. a.

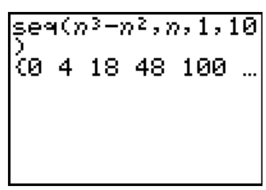


0, 1, 2, 3, 4, 5, 6, 7, 8, 9

b.

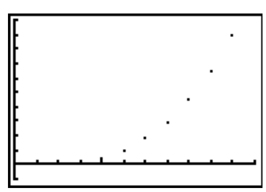


54. a.

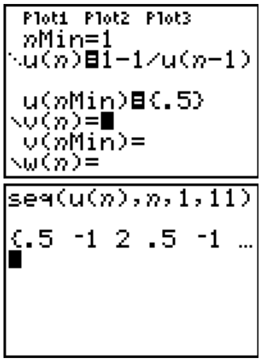


0, 4, 18, 48, 100, 180, 294, 448, 648, 900

b.

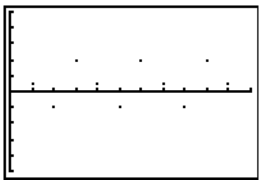


55. a.

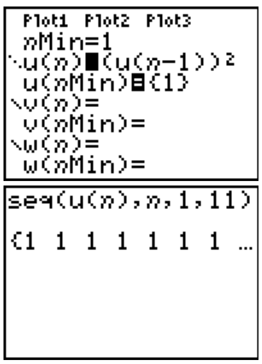


0.5, −1, 2, 0.5, −1, 2, 0.5, −1, 2, 0.5

b.

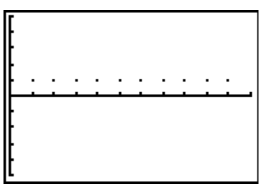


56. a.



1, 1, 1, 1, 1, 1, 1, 1, 1, 1

b.



57. $\dfrac{3!}{5!}=\dfrac{3!}{5\cdot 4\cdot 3!}=\dfrac{1}{20}$

58. $\dfrac{8!}{10!}=\dfrac{8!}{10\cdot 9\cdot 8!}=\dfrac{1}{90}$

59. $\dfrac{12!}{11!}=\dfrac{12\cdot 11!}{11!}=12$

60. $\dfrac{20!}{18!}=\dfrac{20\cdot 19\cdot 18!}{18!}=380$

61. $\dfrac{n!}{(n+1)!}=\dfrac{n!}{(n+1)\cdot n!}=\dfrac{1}{n+1}$

62. $\dfrac{(n-1)!}{(n-2)!}=\dfrac{(n-1)\cdot(n-2)!}{(n-2)!}=n-1$

63. $\dfrac{(2n+1)!}{(2n)!}=\dfrac{(2n+1)\cdot(2n)!}{(2n)!}=2n+1$

64. $\dfrac{(2n+1)!}{(2n-1)!}=\dfrac{(2n+1)(2n)(2n-1)!}{(2n-1)!}=4n^2+2n$

65. $\sum_{k=1}^{7}5=5+5+5+5+5+5+5=35$

66. $\sum_{j=1}^{4}12=12+12+12+12=48$

67. $\sum_{j=0}^{5}j^2=0^2+1^2+2^2+3^2+4^2+5^2=55$

68. $\sum_{k=0}^{4}k^3=0^3+1^3+2^3+3^3+4^3=100$

69.
$$\sum_{i=1}^{5}(2i-1)=(2(1)-1)+(2(2)-1)+(2(3)-1)+(2(4)-1)+(2(5)-1)=25$$

70.
$$\begin{aligned}\sum_{k=0}^{6}(1-3k)&=(1-3(0))+(1-3(1))+(1-3(2))+(1-3(3))\\&\quad+(1-3(4))+(1-3(5))+(1-3(6))\\&=-56\end{aligned}$$

71. $\sum_{j=3}^{7}\dfrac{j+1}{j}=\dfrac{4}{3}+\dfrac{5}{4}+\dfrac{6}{5}+\dfrac{7}{6}+\dfrac{8}{7}=\dfrac{853}{140}$

72.
$$\begin{aligned}\sum_{k=3}^{8}\frac{1}{k+1}&=\frac{1}{3+1}+\frac{1}{4+1}+\frac{1}{5+1}+\frac{1}{6+1}+\frac{1}{7+1}+\frac{1}{8+1}\\&=\frac{2509}{2520}\end{aligned}$$

73.
$$\begin{aligned}\sum_{i=2}^{6}(-1)^i 3^{i-1}&=(-1)^2(3^1)+(-1)^3(3^2)+(-1)^4(3^3)\\&\quad+(-1)^5(3^4)+(-1)^6(3^5)\\&=183\end{aligned}$$

74.
$$\begin{aligned}\sum_{k=2}^{4}(-1)^{k+1}k&=(-1)^{2+1}(2)+(-1)^{3+1}(3)+(-1)^{4+1}(4)=-3\end{aligned}$$

75. $\sum_{k=4}^{7}(2-k^2)$
$= (2-4^2)+(2-5^2)+(2-6^2)+(2-7^2)$
$= -118$

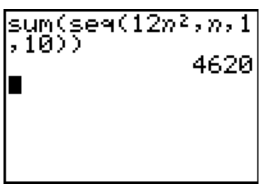
76. $\sum_{j=4}^{9}(j^3+1)$
$= (4^3+1)+(5^3+1)+(6^3+1)+(7^3+1)$
$+(8^3+1)+(9^3+1) = 1995$

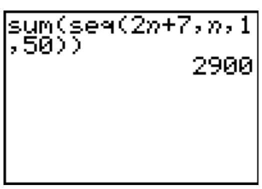
77. $\sum_{k=1}^{51} 2k-1$ **78.** $\sum_{k=1}^{51} 2k$

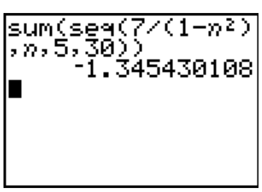
79. $\sum_{k=1}^{11} \frac{1}{5k}$ **80.** $\sum_{k=1}^{9} \frac{2}{k(k+1)}$

81. $\sum_{k=1}^{50} \frac{(-1)^{k+1}}{k}$ **82.** $\sum_{k=1}^{50} (-1)^{k+1}k$

83. $\sum_{k=1}^{10} \frac{k}{k+1}$ **84.** $\sum_{k=1}^{10} \frac{2^k}{k}$

85.


86.


87.


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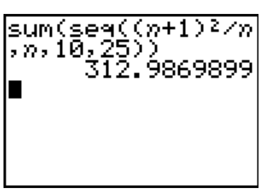
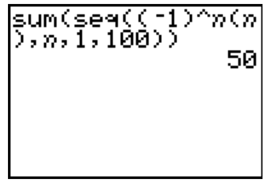


89.


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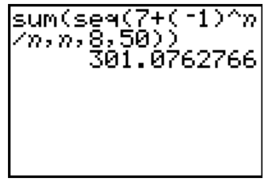


11.1 Applying the Concepts

91. a. Use a table to determine the pattern. It appears that each second, the body falls 32 feet per second faster than it did in the second before.

Time	Distance each second
1	16
2	48
3	80
4	112
5	144
6	176
7	208

b. $16+32(n-1) = 32n-16$ feet

92. a. $1000 \cdot 2^2 = 4000$

b. $1000 \cdot 2^5 = 32,000$ **c.** $1000 \cdot 2^n$

93. The fine is \$50 plus \$25 for each additional day the work is not done. So on the ninth day, she will be fined \$50 + 8(\$25) = \$250.

94. a. Each year the painting appreciates \$40 less than the year before. In the seventh year, the painting appreciates \$1280 – 6(\$40) = \$1040.

b. In the tenth year, the painting appreciates \$1280 – 9(\$40) = \$920.

95. In three years, there are 6 six-month periods. For the first six-month period, cell phone usage was 600 minutes per month. For the next 5 six-month periods, the usage doubled during each period. At the end of three years, cell phone use was $600 \cdot 2^5 = 19,200$ minutes per month.

96. a. $10+7(20)=150$ yards

b. $10+20(n-1)=(20n-10)$ yards

97. a. $A_1=10,000\left(1+\frac{0.06}{2}\right)^1=\$10,300$

$A_2=10,000\left(1+\frac{0.06}{2}\right)^2=\$10,609$

$A_3=10,000\left(1+\frac{0.06}{2}\right)^3=\$10,927.27$

$A_4=10,000\left(1+\frac{0.06}{2}\right)^4=\$11,255.09$

$A_5=10,000\left(1+\frac{0.06}{2}\right)^5=\$11,592.74$

$A_6=10,000\left(1+\frac{0.06}{2}\right)^6=\$11,940.52$

b. The interest is compounded semiannually, so in eight years, there are 16 compounding periods.

$A_{16}=100\left(1+\frac{0.08}{4}\right)^{16}=\$16,047.10$

98. a. $A_1=100\left(1+\frac{0.08}{4}\right)^1=\$102$

$A_2=100\left(1+\frac{0.08}{4}\right)^2=\$104.04$

$A_3=100\left(1+\frac{0.08}{4}\right)^3=\$106.12$

$A_4=100\left(1+\frac{0.08}{4}\right)^4=\$108.24$

$A_5=100\left(1+\frac{0.08}{4}\right)^5=\$110.41$

$A_6=100\left(1+\frac{0.08}{4}\right)^6=\$112.62$

b. The interest is compounded quarterly, so in ten years, there are 40 compounding periods. $A_{40}=100\left(1+\frac{0.08}{4}\right)^{40}=\$220.80$

99. $A_1=(100,000)(1.05)=\$105,000$
$A_2=(105,000)(1.05)=\$110,250$
$A_3=(110,250)(1.05)=\$115,762.50$
$A_4=(115,762.50)(1.05)=\$121,550.63$
$A_5=(121,550.63)(1.05)=\$127,628.16$
$A_6=(127,628.16)(1.05)=\$134,009.56$
$A_7=(134,009.56)(1.05)=\$140,710.04$

The formula is $100,000\left(1.05^n\right)$.

100. $a_1=50,000(0.9)=\$45,000$
$a_2=45,000(0.9^2)=\$40,500$
$a_3=40,500(0.9^3)=\$36,450$
$a_4=36,450(0.9^4)=\$32,805$
$a_5=32,805(0.9^5)=\$29,524.50$
$a_6=29,524.50(0.9^6)=\$26,572.05$
$a_7=26,572.05(0.9^7)=\$23,914.85$
$a_8=23,914.85(0.9^8)=\$21,523.36$
$a_9=21,523.36(0.9^9)=\$19,371.02$
$a_{10}=19,371.02(0.9^{10})=\$17,433.92$

The formula is $50,000\left(0.9^n\right)$.

11.1 Beyond the Basics

101. $a_1=2^{1/2}, a_2=\left(2\cdot 2^{1/2}\right)^{1/2}=2^{3/4},$

$a_3=\left(2\cdot 2^{3/4}\right)^{1/2}=2^{7/8},$

$a_4=\left(2\cdot 2^{7/8}\right)^{1/2}=2^{15/16},$

$a_5=\left(2\cdot 2^{15/16}\right)^{1/2}=2^{31/32}$

$a_n=2^{(2^n-1)/2^n}=2^{1-1/2^n}$

102. a. $a_1=3, a_{n+1}=a_n+2$

b. $b_1=3, b_{n+1}=b_n+1$

103. a. $a_1=1, a_2=\frac{1}{2}, a_3=\frac{1}{4}, a_4=\frac{1}{8}, a_5=\frac{1}{16}$

b. $a_n=\frac{1}{2^{n-1}}$

104. Write the first five terms to determine the pattern: $a_1=\pi\cdot 1^2=\pi, a_2=\pi\cdot(2\cdot 1)^2=4\pi,$

$a_3=\pi\cdot(2\cdot 2)^2=16\pi,$

$a_4=\pi\cdot(2\cdot 4)^2=64\pi,$

$a_5=\pi\cdot(2\cdot 8)^2=256\pi \Rightarrow a_{n+1}=4a_n$

105. $a_1 = a_2 = 1,$
$a_3 = a_{a_2} + a_{3-a_2} = a_1 + a_2 = 1+1 = 2,$
$a_4 = a_{a_3} + a_{4-a_3} = a_2 + a_2 = 1+1 = 2,$
$a_5 = a_{a_4} + a_{5-a_4} = a_2 + a_3 = 1+2 = 3,$
$a_6 = a_{a_5} + a_{6-a_5} = a_3 + a_3 = 2+2 = 4,$
$a_7 = a_{a_6} + a_{7-a_6} = a_4 + a_3 = 2+2 = 4,$
$a_8 = a_{a_7} + a_{8-a_7} = a_4 + a_4 = 2+2 = 4,$
$a_9 = a_{a_8} + a_{9-a_8} = a_4 + a_5 = 2+3 = 5,$
$a_{10} = a_{a_9} + a_{10-a_9} = a_5 + a_5 = 3+3 = 6$

106. $a_0 = 1, a_1 = 3$. Use the formula for a_1 to find a_2: $(a_1)^2 = (-1)^1 a_0 + a_2 \Rightarrow 3^2 = -1 + a_2 \Rightarrow$ $a_2 = 10$. Use the formula again to find a_3: $(a_2)^2 = (-1)^2 a_1 + a_3 \Rightarrow 10^2 = 3 + a_3 \Rightarrow a_3 = 97$

107. $\sum_{n=1}^{20} n^2 = \sum_{m=0}^{19} a_m$

Write the first five terms to determine the pattern: $\sum_{n=1}^{1} n^2 = 1 = \sum_{m=0}^{0} a_m = a_0 \Rightarrow a_0 = 1,$

$\sum_{n=1}^{2} n^2 = 1 + 2^2 = 5 = \sum_{m=0}^{1} a_m = a_0 + a_1 \Rightarrow$
$5 = 1 + a_1 \Rightarrow a_1 = 4 = 2^2.$

$\sum_{n=1}^{3} n^2 = 1 + 2^2 + 3^2 = 14$
$= \sum_{m=0}^{2} a_m = a_0 + a_1 + a_2 \Rightarrow$
$14 = 1 + 4 + a_2 \Rightarrow a_2 = 9 = 3^2$

$\sum_{n=1}^{4} n^2 = 1 + 2^2 + 3^2 + 4^2 = 30$
$= \sum_{m=0}^{3} a_m = a_0 + a_1 + a_2 + a_3 \Rightarrow$
$30 = 1 + 4 + 9 + a_3 \Rightarrow a_3 = 16 = 4^2$

$\sum_{n=1}^{5} n^2 = 1 + 2^2 + 3^2 + 4^2 + 5^2 = 55$
$= \sum_{m=0}^{4} a_m = a_0 + a_1 + a_2 + a_3 + a_4 \Rightarrow$
$55 = 1 + 4 + 9 + 16 + a_4 \Rightarrow a_4 = 25 = 5^2$

Following the pattern, we see that $a_m = (m+1)^2$.

108. Using the summation properties, we have
$\sum_{n=1}^{50} (n-3)^2 = \sum_{n=1}^{50} (n^2 - 6n) + \sum_{n=1}^{50} c \Rightarrow$
$(n-3)^2 = n^2 - 6n + c \Rightarrow$
$n^2 - 6n + 9 = n^2 - 6n + c \Rightarrow 9 = c$

109. $\sum_{n=0}^{10} n^3 = \sum_{m=p}^{q} (m-2)^3 \Rightarrow n^3 = (m-2)^3 \Rightarrow$
$n = m - 2 \Rightarrow m = n + 2$. So, $p = 2$ and $q = 12$.

110. Answers may vary. Sample answer: If $a_k = 1$ and $b_k = 1$, $\sum_{k=1}^{20} (a_k b_k) = 20$ while $\left(\sum_{k=1}^{20} a_k\right)\left(\sum_{k=1}^{20} b_k\right) = 20 \cdot 20 = 400$.

111. $7! - 2(5!) = (7 \cdot 6)(5!) - 2(5!)$
$= (42 - 2)(5!) = 40(5!)$

112.
$$\frac{10!}{6!4!} + \frac{10!}{5!5!} = \frac{10!(5!5!) + 10!(6!4!)}{6!4!5!5!}$$
$$= \frac{10!(5!5! + 6!4!)}{6!4!5!5!} \quad \text{Use the distributive property}$$
$$= \frac{10!5!(5! + 6 \cdot 4!)}{6!4!5!5!} \quad 6! = 6 \cdot 5!$$
$$= \frac{10!5!4!(5+6)}{6!4!5!5!} \quad 5! = 5 \cdot 4!$$
$$= \frac{11 \cdot 10!}{6!5!} = \frac{11!}{6!5!}$$

113. $\frac{1}{5!} + \frac{1}{6!} = \frac{x}{7!} \Rightarrow \frac{1}{5!} + \frac{1}{6 \cdot 5!} = \frac{x}{7 \cdot 6 \cdot 5!}$
Multiply both sides of the equation by $7! = 7 \cdot 6 \cdot 5!$ and simplify. Then solve for x.
$\frac{7 \cdot 6 \cdot 5!}{5!} + \frac{7 \cdot 6 \cdot 5!}{6 \cdot 5!} = \frac{7 \cdot 6 \cdot 5! x}{7 \cdot 6 \cdot 5!} \Rightarrow 42 + 7 = x \Rightarrow$
$x = 49$

114.
$$(n+1)! = 12((n-1)!)$$
$$(n+1)(n)((n-1)!) = 12((n-1)!)$$
$$n^2 + n = 12$$
$$n^2 + n - 12 = 0 \Rightarrow (n-3)(n+4) = 0 \Rightarrow$$
$$n = 3,\ n = -4$$
Since $n!$ is defined only for $n \ge 0$, disregard the negative solution. Thus, $n = 3$.

11.1 Critical Thinking/Discussion/Writing

115. $n^2 + n = (n-1)(n-2)(n-3)(n-4)(n-5) + n^2 + n \Rightarrow 0 = (n-1)(n-2)(n-3)(n-4)(n-5) \Rightarrow n = 1, 2, 3, 4, 5$, so $k = 5$ is the upper limit.

116. If $a_0 = 13$, then $a_1 = 3(13) + 1 = 40,\ a_2 = a_1/2 = 40/2 = 20, a_3 = a_2/2 = 20/2 = 10,\ a_4 = a_3/2 = 10/2 = 5,$
$a_5 = 3a_4 + 1 = 3(5) + 1 = 16,\ a_6 = a_5/2 = 16/2 = 8, a_7 = a_6/2 = 8/2 = 4,\ a_8 = a_7/2 = 4/2 = 2,.$
$a_9 = a_8/2 = 2/2 = 1$
So the conjecture is true for $a_0 = 13$.

117. $\dfrac{(2n)!}{n!} = \dfrac{(2n)(2n-1)(2n-2)(2n-3)(2n-4)(2n-5)\cdots}{n(n-1)(n-2)\cdots} = \dfrac{2n(2n-1)2(n-1)(2n-3)2(n-2)(2n-5)\cdots}{n(n-1)(n-2)\cdots}$
$= 2^n(2n-1)(2n-3)(2n-5)\cdots$
Note that the odd numbers are represented by $2n - 1$, $2n - 3$, $2n - 5$, etc. Therefore,
$\dfrac{(2n)!}{n!} = 2^n(2n-1)(2n-3)(2n-5)\cdots = 2^n\left[1\cdot 3\cdot 5\cdot(2n-1)\right].$

118. $n!$ is divisible by all integers between 2 and n (from the definition of $n!$). So, if we divide $(n! + 1)$ by any integer between 2 and n, we will end up with remainder 1. Hence, $(n! + 1)$ is not divisible by any integer between 2 and n.

11.1 Maintaining Skills

119. $a_2 - a_1 = 5 - 1 = 4$
$a_3 - a_2 = 9 - 5 = 4$
$a_4 - a_3 = 13 - 9 = 4$
$a_5 - a_4 = 17 - 13 = 4$

120. $a_2 - a_1 = 5.2 - 2 = 3.2$
$a_3 - a_2 = 8.4 - 5.2 = 3.2$
$a_4 - a_3 = 11.6 - 8.4 = 3.2$
$a_5 - a_4 = 14.8 - 11.6 = 3.2$

121. $a_2 - a_1 = 1 - 6 = -5$
$a_3 - a_2 = -4 - 1 = -5$
$a_4 - a_3 = -9 - (-4) = -5$
$a_5 - a_4 = -14 - (-9) = -5$

122. $a_2 - a_1 = 1.5 - 5 = -3.5$
$a_3 - a_2 = -2 - 1.5 = -3.5$
$a_4 - a_3 = -5.5 - (-2) = -3.5$
$a_5 - a_4 = -9 - (-5.5) = -3.5$

123. $a_2 = a_1 + d = 3 + 4 = 7$
$a_3 = a_2 + d = 7 + 4 = 11$
$a_4 = a_3 + d = 11 + 4 = 15$
$a_5 = a_4 + d = 15 + 4 = 19$

124. $a_2 = a_1 + d = 4 + 2.5 = 6.5$
$a_3 = a_2 + d = 6.5 + 2.5 = 9$
$a_4 = a_3 + d = 9 + 2.5 = 11.5$
$a_5 = a_4 + d = 11.5 + 2.5 = 14$

125. $a_2 = a_1 + d = 7 + (-3) = 4$
$a_3 = a_2 + d = 4 + (-3) = 1$
$a_4 = a_3 + d = 1 + (-3) = -2$
$a_5 = a_4 + d = -2 + (-3) = -5$

126. $a_2 = a_1 + d = 8 + (-4.5) = 3.5$
$a_3 = a_2 + d = 3.5 + (-4.5) = -1$
$a_4 = a_3 + d = -1 + (-4.5) = -5.5$
$a_5 = a_4 + d = -5.5 + (-4.5) = -10$

127. $a_n = 3n + 5$
$a_1 = 3(1) + 5 = 8$
$a_2 = 3(2) + 5 = 11$
$a_3 = 3(3) + 5 = 14$
$a_4 = 3(4) + 5 = 17$
$a_5 = 3(5) + 5 = 20$
$a_2 - a_1 = 11 - 8 = 3$
$a_3 - a_2 = 14 - 11 = 3$
$a_4 - a_3 = 17 - 14 = 3$
$a_5 - a_4 = 20 - 17 = 3$
Note that
$$\begin{aligned} a_n - a_{n-1} &= (3n+5) - (3(n-1)+1) \\ &= (3n+5) - (3n-3+1) \\ &= (3n+5) - (3n-2) = 3 \end{aligned}$$

128. $a_n = 5n - 7$
$a_1 = 5(1) - 7 = -2$
$a_2 = 5(2) - 7 = 3$
$a_3 = 5(3) - 7 = 8$
$a_4 = 5(4) - 7 = 13$
$a_5 = 5(5) - 7 = 18$
$a_2 - a_1 = 3 - (-2) = 5$
$a_3 - a_2 = 8 - 3 = 5$
$a_4 - a_3 = 13 - 8 = 5$
$a_5 - a_4 = 18 - 13 = 5$
Note that
$$\begin{aligned} a_n - a_{n-1} &= (5n-7) - (5(n-1)-7) \\ &= (5n-7) - (5n-5-7) \\ &= (5n-7) - (5n-12) = 5 \end{aligned}$$

129. $a_n = -2n - 6$
$a_1 = -2(1) - 6 = -8$
$a_2 = -2(2) - 6 = -10$
$a_3 = -2(3) - 6 = -12$
$a_4 = -2(4) - 6 = -14$
$a_5 = -2(5) - 6 = -16$
$a_2 - a_1 = -10 - (-8) = -2$
$a_3 - a_2 = -12 - (-10) = -2$
$a_4 - a_3 = -14 - (-12) = -2$
$a_5 - a_4 = -16 - (-14) = -2$
Note that
$$\begin{aligned} a_n - a_{n-1} &= (-2n-6) - (-2(n-1)-6) \\ &= (-2n-6) - (-2n+2-6) \\ &= (-2n-6) - (-2n-4) = -2 \end{aligned}$$

130. $a_n = -3n + 8$
$a_1 = -3(1) + 8 = 5$
$a_2 = -3(2) + 8 = 2$
$a_3 = -3(3) + 8 = -1$
$a_4 = -3(4) + 8 = -4$
$a_5 = -3(5) + 8 = -7$
$a_2 - a_1 = 2 - 5 = -3$
$a_3 - a_2 = -1 - 2 = -3$
$a_4 - a_3 = -4 - (-1) = -3$
$a_5 - a_4 = -7 - (-4) = -3$
Note that
$$\begin{aligned} a_n - a_{n-1} &= (-3n+8) - (-3(n-1)+8) \\ &= (-3n+8) - (-3n+3+8) \\ &= (-3n+8) - (-3n+11) = -3 \end{aligned}$$

131. $a_n = 2n - 5$
$a_{n+1} = 2(n+1) - 5 = 2n + 2 - 5 = 2n - 3$
$a_{n-1} = 2(n-1) - 5 = 2n - 2 - 5 = 2n - 7$

132. $a_n = -3n + 4$
$a_{n+1} = -3(n+1) + 4 = -3n - 3 + 4 = -3n + 1$
$a_{n-1} = -3(n-1) + 4 = -3n + 3 + 4 = -3n + 7$

11.2 Arithmetic Sequences; Partial Sums

11.2 Practice Problems

1. (−2) − 3 = −5, −7 −(−2) = −5, etc. The common difference is −5.

2. $a_1 = -3$ and d = 1 − (−3) = 4, so the expression is
$a_n = -3 + 4(n-1) = -3 + 4n - 4 = 4n - 7.$

3. $a_4 = a_1 + d(4-1) \Rightarrow 41 = a_1 + 3d$
$a_{15} = a_1 + d(15-1) \Rightarrow 8 = a_1 + 14d$
$$\begin{cases} a_1 + 3d = 41 \\ a_1 + 14d = 8 \end{cases} \Rightarrow -11d = 33 \Rightarrow d = -3, a_1 = 50$$
$a_n = 50 - 3(n-1) = -3n + 53$

4. $d = \frac{5}{6} - \frac{2}{3} = \frac{1}{6};\ n = 10;\ a_1 = \frac{2}{3};\ a_{10} = \frac{13}{6}$
$$S_{10} = 10\left(\frac{\frac{2}{3} + \frac{13}{6}}{2}\right) = \frac{85}{6}$$

5. $d = 32;\ n = 5;\ a_{11} = 32(11) - 16 = 336;$
$a_{15} = 32(15) - 16 = 464$
$$S = 5\left(\frac{336 + 464}{2}\right) = 2000$$
The object fell 2000 feet during the 11th through 15th seconds.

11.2 Basic Concepts and Skills

1. If 5 is the common difference of an arithmetic sequence with general term a_n, then $a_{17} - a_{16} = \underline{5}$.

2. If 5 is the common difference of an arithmetic sequence with a general term a_n and $a_{21} = 7$, then $a_{22} = \underline{12}$.

3. If 14 is the term immediately following the sequence term 17 is an arithmetic sequence, then the common difference is $\underline{-3}$.

4. If $a_1 = 2$ and $a_{11} = 22$ for an arithmetic sequence, then the common difference, d, is $\underline{2}$.

5. False. The common difference of an arithmetic sequence may be positive or negative.

6. True

7. The sequence is arithmetic. $a_1 = 1, d = 1$

8. The sequence is arithmetic. $a_1 = 1, d = 2$

9. The sequence is arithmetic. $a_1 = 2, d = 3$

10. The sequence is arithmetic. $a_1 = 10, d = -3$

11. The sequence is not arithmetic.

12. The sequence is not arithmetic.

13. The sequence is not arithmetic.

14. The sequence is arithmetic. $a_1 = -\frac{1}{4}, d = \frac{1}{2}$

15. The sequence is arithmetic. $a_1 = 0.6, d = -0.4$

16. The sequence is arithmetic. $a_1 = 2.3, d = 0.4$

17. Write the first four terms of the sequence:
$a_1 = 2(1) + 6 = 8, a_2 = 2(2) + 6 = 10,$
$a_3 = 2(3) + 6 = 12, a_4 = 2(4) + 6 = 14$.
It appears that the sequence is arithmetic, with a common difference of 2. Verify as follows:
$d = a_{n+1} - a_n = (2(n+1) + 6) - (2n + 6) = 2$.

18. Write the first four terms of the sequence:
$a_1 = 1 - 5(1) = -4, a_2 = 1 - 5(2) = -9,$
$a_3 = 1 - 5(3) = -14, a_4 = 1 - 5(4) = -19$.
It appears that the sequence is arithmetic, with a common difference of -5. Verify as follows:
$d = a_{n+1} - a_n = (1 - 5(n+1)) - (1 - 5n) = -5$.

19. Write the first four terms of the sequence:
$a_1 = 1 - 1^2 = 0, a_2 = 1 - 2^2 = -3,$
$a_3 = 1 - 3^2 = -8, a_4 = 1 - 4^2 = -15$.
There is no common difference, so the sequence is not arithmetic.

20. Write the first four terms of the sequence:
$a_1 = 2(1^2) - 3 = -1, a_2 = 2(2^2) - 3 = 5,$
$a_3 = 2(3^2) - 3 = 15, a_4 = 2(4^2) - 3 = 29$.
There is no common difference, so the sequence is not arithmetic.

21. $d = 3, a_1 = 5 \Rightarrow a_n = 5 + 3(n-1) = 3n + 2$

22. $d = 3, a_1 = 4 \Rightarrow a_n = 4 + 3(n-1) = 3n + 1$

23. $d = -5, a_1 = 11 \Rightarrow a_n = 11 - 5(n-1) = 16 - 5n$

24. $d = -4, a_1 = 9 \Rightarrow a_n = 9 - 4(n-1) = 13 - 4n$

25. $d = -\frac{1}{4}, a_1 = \frac{1}{2} \Rightarrow a_n = \frac{1}{2} - \frac{1}{4}(n-1) = \frac{3-n}{4}$

26. $d = \frac{1}{6}, a_1 = \frac{2}{3} \Rightarrow$
$a_n = \frac{2}{3} + \frac{1}{6}(n-1) = \frac{1}{2} + \frac{n}{6} = \frac{n+3}{6}$

27. $d = -\frac{2}{5}, a_1 = -\frac{3}{5} \Rightarrow$
$a_n = -\frac{3}{5} - \frac{2}{5}(n-1) = \frac{-1-2n}{5} = -\frac{2n+1}{5}$

28. $d = \frac{3}{2}, a_1 = \frac{1}{2} \Rightarrow a_n = \frac{1}{2} + \frac{3}{2}(n-1) = \frac{3n-2}{2}$

29. $d = 3, a_1 = e \Rightarrow a_n = e + 3(n-1) = e + 3n - 3$

30. $d = 4, a_1 = 2\pi \Rightarrow$
$a_n = 2\pi + 4(n-1) = 4n + 2\pi - 4$

31. $a_4 = a_1 + d(4-1) \Rightarrow 21 = a_1 + 3d$
$a_{10} = a_1 + d(10-1) \Rightarrow 60 = a_1 + 9d$
$\begin{cases} a_1 + 3d = 21 \\ a_1 + 9d = 60 \end{cases} \Rightarrow -6d = -39 \Rightarrow d = \frac{13}{2}, a_1 = \frac{3}{2}$
$a_n = \frac{3}{2} + \frac{13}{2}(n-1) = \frac{13}{2}n - 5$

32. $a_3 = a_1 + d(3-1) \Rightarrow 15 = a_1 + 2d$
$a_{21} = a_1 + d(21-1) \Rightarrow 87 = a_1 + 20d$
$\begin{cases} a_1 + 2d = 15 \\ a_1 + 20d = 87 \end{cases} \Rightarrow 18d = 72 \Rightarrow d = 4, a_1 = 7$
$a_n = 7 + 4(n-1) = 4n + 3$

33. $a_7 = a_1 + d(7-1) \Rightarrow 8 = a_1 + 6d$
$a_{15} = a_1 + d(15-1) \Rightarrow -8 = a_1 + 14d$
$\begin{cases} a_1 + \ \ 6d = 8 \\ a_1 + 14d = -8 \end{cases} \Rightarrow 8d = -16 \Rightarrow d = -2, a_1 = 20$
$a_n = 20 - 2(n-1) = 22 - 2n$

34. $a_5 = a_1 + d(5-1) \Rightarrow 12 = a_5 + 4d$
$a_{18} = a_1 + d(18-1) \Rightarrow -1 = a_1 + 17d$
$\begin{cases} a_1 + \ \ 4d = 12 \\ a_1 + 17d = -1 \end{cases} \Rightarrow 13d = -13 \Rightarrow d = -1, a_1 = 16$
$a_n = 16 - (n-1) = 17 - n$

35. $a_3 = a_1 + d(3-1) \Rightarrow 7 = a_1 + 2d$
$a_{23} = a_1 + d(23-1) \Rightarrow 17 = a_1 + 22d$
$\begin{cases} a_1 + \ \ 2d = 7 \\ a_1 + 22d = 17 \end{cases} \Rightarrow 20d = 10 \Rightarrow d = \frac{1}{2}, a_1 = 6$
$a_n = 6 + \frac{1}{2}(n-1) = \frac{n+11}{2}$

36. $a_{11} = a_1 + d(11-1) \Rightarrow -1 = a_1 + 10d$
$a_{31} = a_1 + d(31-1) \Rightarrow 5 = a_1 + 30d$
$\begin{cases} a_1 + 10d = -1 \\ a_1 + 30d = 5 \end{cases} \Rightarrow 20d = 6 \Rightarrow d = \dfrac{3}{10}, a_1 = -4$
$a_n = -4 + \dfrac{3}{10}(n-1) = \dfrac{3n-43}{10}$

37. $n = 50; S = 50\left(\dfrac{1+50}{2}\right) = 1275$

38. $d = 2, a_1 = 2, a_n = 102 \Rightarrow 102 = 2 + 2(n-1) \Rightarrow$
$n = 51; S = 51\left(\dfrac{2+102}{2}\right) = 2652$

39. $d = 2, a_1 = 1, a_n = 99 \Rightarrow 99 = 1 + 2(n-1) \Rightarrow$
$n = 50; S = 50\left(\dfrac{1+99}{2}\right) = 2500$

40. $d = 5, a_1 = 5, a_n = 200 \Rightarrow 200 = 5 + 5(n-1) \Rightarrow$
$n = 40; S = 40\left(\dfrac{5+200}{2}\right) = 4100$

41. $d = 3, a_1 = 3, a_n = 300 \Rightarrow 300 = 3 + 3(n-1) \Rightarrow$
$n = 100; S = 100\left(\dfrac{3+300}{2}\right) = 15{,}150$

42. $d = 3, a_1 = 4, a_n = 301 \Rightarrow 301 = 4 + 3(n-1) = a_n$
$n = 100; S = 100\left(\dfrac{4+301}{2}\right) = 15{,}250$

43. $d = -3, a_1 = 2, a_n = -34 \Rightarrow$
$-34 = 2 - 3(n-1) \Rightarrow n = 13$
$S = 13\left(\dfrac{2-34}{2}\right) = -208$

44. $d = -5, a_1 = -3, a_n = -48 \Rightarrow$
$-48 = -3 - 5(n-1) \Rightarrow n = 10$
$S = 10\left(\dfrac{-3-48}{2}\right) = -255$

45. $d = \dfrac{2}{3}, a_1 = \dfrac{1}{3}, a_n = 7 \Rightarrow$
$7 = \dfrac{1}{3} + \dfrac{2}{3}(n-1) \Rightarrow n = 11$
$S = 11\left(\dfrac{1/3+7}{2}\right) = \dfrac{121}{3}$

46. $d = \dfrac{7}{5}, a_1 = \dfrac{3}{5}, a_n = \dfrac{101}{5} \Rightarrow$
$\dfrac{101}{5} = \dfrac{3}{5} + \dfrac{7}{5}(n-1) \Rightarrow n = 15$
$S = 15\left(\dfrac{3/5+101/5}{2}\right) = 156$

47. $d = 5, n = 50, a_1 = 2 \Rightarrow a_n = 2 + 5(50-1) = 247$
$S = 50\left(\dfrac{2+247}{2}\right) = 6225$

48. $d = 2, n = 40, a_1 = 8 \Rightarrow a_n = 8 + 2(40-1) = 86$
$S = 40\left(\dfrac{8+86}{2}\right) = 1880$

49. $d = 4, n = 20, a_1 = -15 \Rightarrow$
$a_n = -15 + 4(20-1) = 61$
$S = 20\left(\dfrac{-15+61}{2}\right) = 460$

50. $d = 7, n = 25, a_1 = -20 \Rightarrow$
$a_n = -20 + 7(25-1) = 148$
$S = 25\left(\dfrac{-20+148}{2}\right) = 1600$

51. $d = 0.2, n = 100, a_1 = 3.5 \Rightarrow$
$a_n = 3.5 + 0.2(100-1) = 23.3$
$S = 100\left(\dfrac{3.5+23.3}{2}\right) = 1340$

52. $d = 0.5, n = 80, a_1 = -7 \Rightarrow$
$a_n = -7 + 0.5(80-1) = 32.5$
$S = 80\left(\dfrac{-7+32.5}{2}\right) = 1020$

In exercises 53−58, use the formula for finding the terms in an arithmetic sequence, $a_n = a_1 + d(n-1)$.

53. $a_n = 75,\ a_1 = 1,\ d = 2$
$75 = 1 + 2(n-1) \Rightarrow 37 = n-1 \Rightarrow n = 38$

54. $a_n = 120,\ a_1 = 2,\ d = 2$
$120 = 2 + 2(n-1) \Rightarrow 59 = n-1 \Rightarrow n = 60$

55. $a_n = 95,\ a_1 = -1,\ d = 4$
$95 = -1 + 4(n-1) \Rightarrow 24 = n-1 \Rightarrow n = 25$

56. $a_n = 83,\ a_1 = -5,\ d = 2$
$83 = -5 + 2(n-1) \Rightarrow 44 = n-1 \Rightarrow n = 45$

57. $a_n = 50\sqrt{3},\ a_1 = 2\sqrt{3},\ d = 2\sqrt{3}$
$50\sqrt{3} = 2\sqrt{3} + 2\sqrt{3}(n-1) \Rightarrow$
$48\sqrt{3} = 2\sqrt{3}(n-1) \Rightarrow 24 = n-1 \Rightarrow n = 25$

58. $a_n = 73\pi,\ a_1 = 3\pi,\ d = 2\pi$
$73\pi = 3\pi + 2\pi(n-1) \Rightarrow 70\pi = 2\pi(n-1) \Rightarrow$
$35 = n-1 \Rightarrow n = 36$

11.2 Applying the Concepts

59. $d = 2, n = 30, a_1 = 10 \Rightarrow a_n = 10 + 2(30-1) = 68$
$S = 30\left(\frac{10+68}{2}\right) = 1170$

60. a. $d = -3, a_1 = 60 \Rightarrow$
$a_n = 60 - 3(n-1) = (63 - 3n)$ feet

b. $a_{15} = 63 - 3(15) = 18;$
$S = 15\left(\frac{60+18}{2}\right) = 585$ ft

61. $d = 25, n = 30, a_1 = 50 \Rightarrow$
$a_{30} = 50 + 25(30-1) = 775$
$S = 30\left(\frac{50+775}{2}\right) = \$12,375$

62. $a_3 = a_1 + d(3-1) \Rightarrow 820 = a_1 + 2d$
$a_{12} = a_1 + d(12-1) \Rightarrow 910 = a_1 + 11d$
$\begin{cases} a_1 + \ \ 2d = 820 \\ a_1 + 11d = 910 \end{cases} \Rightarrow 9d = 90 \Rightarrow d = 10, a_1 = 800$
$S = 12\left(\frac{800+910}{2}\right) = \$10,260$

63. Antonio has 17 different hourly wages over the four years (his original wage plus 16 raises), so $n = 17$. $a_1 = \$12.75, d = 0.25 \Rightarrow$
$a_{13} = 12.75 + 0.25(17-1) = \$16.75$.

64. For the marketing company,
$a_1 = \$32,500, d = \$1300, n = 5 \Rightarrow$
$a_5 = 32,500 + (5-1)(1300) = \$37,700$
$S = 5\left(\frac{32,500+37,700}{2}\right) = \$175,500$
For the exporting company,
$a_1 = \$36,000, d = \$400, n = 5 \Rightarrow$
$a_5 = 36,000 + (5-1)(400) = \$37,600$
$S = 5\left(\frac{36,000+37,600}{2}\right) = \$184,000$
The exporting company will pay \$8500 more than the marketing company.

65. $n = 25, a_1 = 20, d = 2 \Rightarrow$
$a_{25} = 20 + 2(25-1) = 68$
$S = 25\left(\frac{20+68}{2}\right) = 1100$.
There are 1100 seats in the theater.

66. $n = 50, a_1 = 16, a_{50} = 65 \Rightarrow$
$S = 50\left(\frac{16+65}{2}\right) = 2025$.
There are 2025 bricks in the driveway.

67. $n = 17, a_1 = 40, a_{28} = 8 \Rightarrow$
$S = 17\left(\frac{40+8}{2}\right) = 408$
There are 408 boxes in the stack.

68. Using the formula for the sum of an arithmetic sequence with $a_1 = 10$, we have
$325 = n\left(\frac{10+a_n}{2}\right) \Rightarrow 650 = 10n + na_n$. The last term in the sequence, given $d = 5$, is
$a_n = 10 + 5(n-1) = 5n + 5$. Solve the system:
$\begin{cases} 10n + na_n = 650 \\ a_n = 5n - 5 \end{cases} \Rightarrow 10n + n(5n+5) = 650 \Rightarrow$
$5n^2 + 15n = 650 \Rightarrow n^2 + 3n - 130 = 0 \Rightarrow$
$(n+13)(n-10) = 0 \Rightarrow n = -13$ (reject this) or
$n = 10$.
Sophia saved for 10 days.

69. Using the formula for the sum of an arithmetic sequence with $a_1 = 3$, we have
$192 = n\left(\frac{3+a_n}{2}\right) \Rightarrow 384 = 3n + na_n$. The last term in the sequence, given $d = 6$, is
$a_n = 3 + 6(n-1) = 6n - 3$. Solve the system:
$\begin{cases} 3n + na_n = 384 \\ a_n = 6n - 3 \end{cases} \Rightarrow 3n + n(6n-3) = 384 \Rightarrow$
$6n^2 = 384 \Rightarrow n^2 = 64 \Rightarrow n = \pm 8$ (Reject the negative solution.) The flea market was open for 8 days.

70. Car *A* travels $60h$ miles, so we want to find out when the distance car *B* traveled $\geq 60h$ where *h* represents the number of hours.
Car *B* travels 50 miles in the first hour,
50 + 55 miles = 105 miles in two hours,
50 + 55 + 60 miles = 165 miles in three hours, etc. Car *B*’s speed is represented by
$(50 + 5(h-1))$ mph. Using the formula for the sum of an arithmetic sequence with $a_1 = 50$, we have
$60h = h\left(\frac{50 + (50 + 5(h-1))}{2}\right) \Rightarrow$
$60h = h\left(\frac{95+5h}{2}\right) \Rightarrow 120h = 95h + 5h^2 \Rightarrow \cdot$
$5h^2 - 25h = 0 \Rightarrow 5h(h-5) = 0 \Rightarrow$
$h = 0,\ h = 5$
$h = 0$ makes no sense in terms of the problem, so we disregard this solution. Car *B* will overtake car *A* after 5 hours.

11.2 Beyond the Basics

71. $a_n = \log\left(\dfrac{a^n}{b^{n-1}}\right),\ a_{n+1} = \log\left(\dfrac{a^{n+1}}{b^n}\right)$

$$a_{n+1} - a_n = \log\left(\frac{a^{n+1}}{b^n}\right) - \log\left(\frac{a^n}{b^{n-1}}\right)$$
$$= \log\left(\frac{\frac{a^{n+1}}{b^n}}{\frac{a^n}{b^{n-1}}}\right) = \log\left(\frac{a^{n+1}}{b^n}\cdot\frac{b^{n-1}}{a^n}\right)$$
$$= \log\frac{a}{b}$$

72. $a_n = \log\left(ab^n\right),\ a_{n+1} = \log\left(ab^{n+1}\right)$

$$a_{n+1} - a_n = \log\left(ab^{n+1}\right) - \log\left(ab^n\right)$$
$$= \log\left(\frac{ab^{n+1}}{ab^n}\right) = \log b$$

73. $S_n = 3n^2 + 4n$

$$S_{n-1} = 3(n-1)^2 + 4(n-1)$$
$$= 3\left(n^2 - 2n + 1\right) + 4n - 4$$
$$= 3n^2 - 6n + 3 + 4n - 4$$
$$= 3n^2 - 2n - 1$$
$$a_n = S_n - S_{n-1} = \left(3n^2 + 4n\right) - \left(3n^2 - 2n - 1\right)$$
$$= 6n + 1$$
$$d = a_n - a_{n-1} = (6n+1) - (6(n-1)+1)$$
$$= (6n+1) - (6n-5) = 6$$

Since $d = 6$ is a constant, the sequence a_n is an arithmetic sequence.

74. Assume that $S_n = An^2 + Bn$. Then

$$S_{n-1} = A(n-1)^2 + B(n-1)$$
$$= A\left(n^2 - 2n + 1\right) + (Bn - B)$$
$$= An^2 - 2An + A + (Bn - B)$$
$$= An^2 - 2An + Bn + A - B$$
$$a_n = S_n - S_{n-1}$$
$$= \left(An^2 + Bn\right) - \left(An^2 - 2An + Bn + A - B\right)$$
$$= 2An - A + B$$
$$d = a_n - a_{n-1}$$
$$= (2An - A + B) - \left[2A(n-1) - A + B\right]$$
$$= (-2An - A - B) - (2An - 2A - A - B)$$
$$= -2An - A - B - 2An + 3A + B$$
$$= 2A$$

Since $d = 2A$ is a constant, the sequence a_n is an arithmetic sequence.

Now we must prove that if sequence a_n is an arithmetic sequence, then the sum S_n of the first n terms is of the form $An^2 + Bn$. We have the formulas

$S_n = n\left(\dfrac{a_1 + a_n}{2}\right)$ and $a_n = a_1 + (n-1)d$. Thus,

$$S_n = n\left(\frac{a_1 + a_1 + (n-1)d}{2}\right)$$
$$= \frac{n}{2}\left(2a_1 + dn - d\right)$$
$$= \frac{d}{2}n^2 + \left(\frac{2a_1 - d}{2}\right)n,$$

where $A = \dfrac{d}{2}$ and $B = \dfrac{2a_1 - d}{2}$.

Therefore, the sequence a_n is an arithmetic sequence if and only if the sum S_n of the first n terms is of the form $An^2 + Bn$.

75. 45 is divisible by 3 and the largest number less than 100 that is divisible by 3 is 99, so $a_1 = 45$, $d = 3, a_n = 99 \Rightarrow 99 = 45 + 3(n-1) \Rightarrow n = 19$.

$$S = 19\left(\frac{45 + 99}{2}\right) = 1368.$$

76. The first multiple of 7 that is greater than 26 is 28 and the largest multiple of 7 that is less than 120 is 119, so $a_1 = 28, a_n = 119, d = 7 \Rightarrow$ $119 = 28 + 7(n-1) \Rightarrow n = 14$

$$S = 14\left(\frac{28 + 119}{2}\right) = 1029$$

77. The sequence of the reciprocals is $2, \frac{5}{3}, \frac{4}{3}, 1, \ldots$. The common difference is $-\frac{1}{3}$, so the sequence is arithmetic, and the original sequence is harmonic.

78. a. The numerators are the same, so find the common difference of the denominators to find the fourth term: $d = 2 \Rightarrow$ the fourth term of the original sequence is $2/11$.

b. Find the nth term of the sequence composed of the denominators: $a_n = 5 + 2(n-1) = 2n + 3$, so the nth term of the original sequence is $\dfrac{2}{2n+3}$.

79. $m = a + d$ and $b = a + 2d$. So

$$\frac{a+b}{2} = \frac{a + a + 2d}{2} = \frac{2a + 2d}{2} = a + d = m$$

11.2 Critical Thinking/Discussion/Writing

80. $a = a_1 = 1, b = a_n = 30 \Rightarrow$
$S = 465 = n\left(\frac{1+30}{2}\right) \Rightarrow n = 30$. So, there are 30 terms in the sequence and $k = 28$.
$30 = 1 + d(30-1) \Rightarrow d = 1$.

81. $a_1 = 10, a_{21} = a_1 + d(21-1) \Rightarrow 0 = 10 + 20d \Rightarrow$
$d = -\frac{1}{2} \Rightarrow a_n = 10 - \frac{1}{2}(n-1) = \frac{21-n}{2}$

82. First term $= -a_1$; difference $= -d$.

83. There are 100 − 22 + 1 = 79 terms in the series.

84. In the set of counting numbers, $a_n = n$.
$12,403 = n\left(\frac{1+n}{2}\right) \Rightarrow 24,806 = n^2 + n \Rightarrow$
$n^2 + n - 24,806 = 0 \Rightarrow (n-157)(n+158) = 0 \Rightarrow$
$n = -158$ (reject this) or $n = 157$.
The sum of the first 157 counting number is 12,403.

11.2 Maintaining Skills

85.
$$\frac{a_2}{a_1} = \frac{6}{3} = 2, \quad \frac{a_3}{a_2} = \frac{12}{6} = 2, \quad \frac{a_4}{a_3} = \frac{24}{12} = 2, \quad \frac{a_5}{a_4} = \frac{48}{24} = 2$$

86.
$$\frac{a_2}{a_1} = \frac{4/3}{2} = \frac{2}{3}, \quad \frac{a_3}{a_2} = \frac{8/9}{4/3} = \frac{2}{3}, \quad \frac{a_4}{a_3} = \frac{16/27}{8/9} = \frac{2}{3}, \quad \frac{a_5}{a_4} = \frac{32/81}{16/27} = \frac{2}{3}$$

87.
$$\frac{a_2}{a_1} = \frac{-12}{4} = -3, \quad \frac{a_3}{a_2} = \frac{36}{-12} = -3, \quad \frac{a_4}{a_3} = \frac{-108}{36} = -3, \quad \frac{a_5}{a_4} = \frac{324}{-108} = -3$$

88.
$$\frac{a_2}{a_1} = \frac{-3/2}{3} = -\frac{1}{2}, \quad \frac{a_3}{a_2} = \frac{3/4}{-3/2} = -\frac{1}{2}, \quad \frac{a_4}{a_3} = \frac{-3/8}{3/4} = -\frac{1}{2}, \quad \frac{a_5}{a_4} = \frac{-3/8}{3/16} = -\frac{1}{2}$$

89.
$a_2 = a_1 r = 2 \cdot 3 = 6$
$a_3 = a_2 r = 6 \cdot 3 = 18$
$a_4 = a_3 r = 18 \cdot 3 = 54$
$a_5 = a_4 r = 54 \cdot 3 = 162$

90.
$a_2 = a_1 r = 3 \cdot \frac{2}{5} = \frac{6}{5}$
$a_3 = a_2 r = \frac{6}{5} \cdot \frac{2}{5} = \frac{12}{25}$
$a_4 = a_3 r = \frac{12}{25} \cdot \frac{2}{5} = \frac{24}{125}$
$a_5 = a_4 r = \frac{24}{125} \cdot \frac{2}{5} = \frac{48}{625}$

91.
$a_2 = a_1 r = 1(-2) = -2$
$a_3 = a_2 r = -2(-2) = 4$
$a_4 = a_3 r = 4(-2) = -8$
$a_5 = a_4 r = -8(-2) = 16$

92.
$a_2 = a_1 r = 1\left(-\frac{3}{2}\right) = -\frac{3}{2}$
$a_3 = a_2 r = \left(-\frac{3}{2}\right)\left(-\frac{3}{2}\right) = \frac{9}{4}$
$a_4 = a_3 r = \frac{9}{4}\left(-\frac{3}{2}\right) = -\frac{27}{8}$
$a_5 = a_4 r = -\frac{27}{8}\left(-\frac{3}{2}\right) = \frac{81}{16}$

93.
$a_n = 5 \cdot 2^n$
$a_1 = 5 \cdot 2^1 = 10$
$a_2 = 5 \cdot 2^2 = 20$
$a_3 = 5 \cdot 2^3 = 40$
$a_4 = 5 \cdot 2^4 = 80$
$a_5 = 5 \cdot 2^5 = 160$

$$\frac{a_2}{a_1} = \frac{20}{10} = 2, \quad \frac{a_3}{a_2} = \frac{40}{20} = 2, \quad \frac{a_4}{a_3} = \frac{80}{40} = 2, \quad \frac{a_5}{a_4} = \frac{160}{80} = 2$$

Note that
$$\frac{a_{n+1}}{a_n} = \frac{5 \cdot 2^{n+1}}{5 \cdot 2^n} = \frac{5 \cdot 2^n \cdot 2}{5 \cdot 2^n} = 2$$

94.
$a_n = 4 \cdot 3^n$
$a_1 = 4 \cdot 3^1 = 12$
$a_2 = 4 \cdot 3^2 = 36$
$a_3 = 4 \cdot 3^3 = 108$
$a_4 = 4 \cdot 3^4 = 324$
$a_5 = 4 \cdot 3^5 = 972$

$$\frac{a_2}{a_1} = \frac{36}{12} = 3, \quad \frac{a_3}{a_2} = \frac{108}{36} = 3, \quad \frac{a_4}{a_3} = \frac{324}{108} = 3, \quad \frac{a_5}{a_4} = \frac{972}{324} = 3$$

Note that
$$\frac{a_{n+1}}{a_n} = \frac{4 \cdot 3^{n+1}}{4 \cdot 3^n} = \frac{4 \cdot 3^n \cdot 3}{4 \cdot 3^n} = 3$$

95. $a_n = 2(-3)^n$

$a_1 = 2(-3)^1 = -6$

$a_2 = 2(-3)^2 = 18$

$a_3 = 2(-3)^3 = -54$

$a_4 = 2(-3)^4 = 162$

$a_5 = 2(-3)^5 = -486$

$\frac{a_2}{a_1} = \frac{18}{-6} = -3$

$\frac{a_3}{a_2} = \frac{-54}{18} = -3$

$\frac{a_4}{a_3} = \frac{162}{-54} = -3$

$\frac{a_5}{a_4} = \frac{-486}{162} = -3$

Note that

$$\frac{a_{n+1}}{a_n} = \frac{2(-3)^{n+1}}{2(-3)^n} = \frac{2(-3)^n(-3)}{2(-3)^n} = -3$$

96. $a_n = 5(-2)^{-n}$

$a_1 = 5(-2)^{-1} = 5\left(-\frac{1}{2}\right) = -\frac{5}{2}$

$a_2 = 5(-2)^{-2} = 5\left(\frac{1}{4}\right) = \frac{5}{4}$

$a_3 = 5(-2)^{-3} = 5\left(-\frac{1}{8}\right) = -\frac{5}{8}$

$a_4 = 5(-2)^{-4} = 5\left(\frac{1}{16}\right) = \frac{5}{16}$

$a_5 = 5(-2)^{-5} = 5\left(-\frac{1}{32}\right) = -\frac{5}{32}$

$\frac{a_2}{a_1} = \frac{5/4}{-5/2} = -\frac{1}{2}$

$\frac{a_3}{a_2} = \frac{-5/8}{5/4} = -\frac{1}{2}$

$\frac{a_4}{a_3} = \frac{5/16}{-5/8} = -\frac{1}{2}$

$\frac{a_5}{a_4} = \frac{5/32}{5/16} = -\frac{1}{2}$

Note that

$$\frac{a_{n+1}}{a_n} = \frac{5(-2)^{-(n+1)}}{5(-2)^{-n}} = \frac{5(-2)^n}{5(-2)^{n+1}} = -\frac{1}{2}$$

97. $a_n = -3\cdot 2^n$

$a_{n+1} = -3\cdot 2^{n+1}$

$a_{n-1} = -3\cdot 2^{n-1}$

98. $a_n = 5(-3)^{-n}$

$a_{n+1} = 5(-3)^{-(n+1)} = 5(-3)^{-n-1}$

$a_{n-1} = 5(-3)^{-(n-1)} = 5(-3)^{1-n}$

11.3 Geometric Sequences and Series

11.3 Practice Problems

1. $r = \frac{18}{6} = 3$

2. The first four terms of the sequence are $\frac{3}{2}, \left(\frac{3}{2}\right)^2 = \frac{9}{4}, \left(\frac{3}{2}\right)^3 = \frac{27}{8}, \left(\frac{3}{2}\right)^4 = \frac{81}{16}.$

Since the common ratio is $\frac{3}{2}$, the sequence is geometric.

3. a. $a_1 = 2$ **b.** $r = \frac{6/5}{2} = \frac{3}{5}$

c. $a_n = 2\cdot\left(\frac{3}{5}\right)^{n-1}$

4. $a_{18} = a_1 r^{18-1} = 7(1.5^{17}) \approx 6896.8288$

5. $a_1 = \frac{1}{9};\ r = \frac{1/3}{1/9} = 3$

$a_n = a_1 r^{n-1} \Rightarrow \frac{1}{9}\cdot 3^{n-1} = 243 \Rightarrow$

$3^{n-1} = 2187 \Rightarrow 3^{n-1} = 3^7 \Rightarrow$

$n - 1 = 7 \Rightarrow n = 8$

There are 8 terms in the sequence.

6. $a_1 = 3(0.4)^1 = 1.2;\ r = 0.4$

$$S_{17} = \sum_{i=1}^{17} 3(0.4)^i = 1.2\left(\frac{1-0.4^{17}}{1-0.4}\right) \approx 2$$

7. $$A = 1500\left[\frac{\left(1+\frac{0.045}{1}\right)^{(1)(30)} - 1}{\frac{0.045}{1}}\right] = \$91,510.60$$

8. $a_1 = 3,\ r = \frac{6/3}{3} = \frac{2}{3}$

Since $|r| < 1$, we can use the formula for the sum of an infinite geometric series.

$$S = \frac{a_1}{1-r} = \frac{3}{1-\frac{2}{3}} = 9$$

9. $\sum_{i=1}^{\infty}(10,000,000)(0.85)^{i-1} = \frac{a_1}{1-r} = \frac{10,000,000}{1-.85} \approx \$66,666,666.67$

11.3 Basic Concepts and Skills

1. If 5 is the common ratio of a geometric sequence with general term a_n, then $\frac{a_{63}}{a_{62}} = \underline{5}$.
2. If 5 is the common ratio of a geometric sequence with general term a_n and $a_{15} = -2$, then $a_{16} = \underline{-10}$.
3. If 24 is the term immediately following the sequence term 8 is a geometric sequence, then the common ratio is $\underline{3}$.
4. An infinite geometric series $a_1 + a_1 r + a_1 r^2 + \cdots$ does not have a sum if $\underline{|r|} \geq 1$.
5. True
6. False.
7. The sequence is geometric. $a_1 = 3, r = 2$
8. The sequence is geometric. $a_1 = 2, r = 2$
9. The sequence is not geometric.
10. The sequence is not geometric.
11. The sequence is geometric. $a_1 = 1, r = -3$
12. The sequence is geometric. $a_1 = -1, r = -2$
13. The sequence is geometric. $a_1 = 7, r = -1$
14. The sequence is geometric. $a_1 = 1, r = -2$
15. The sequence is geometric. $a_1 = 9, r = \frac{1}{3}$
16. The sequence is geometric. $a_1 = 5, r = \frac{2}{5}$
17. The sequence is geometric. $a_1 = -\frac{1}{2}, r = -\frac{1}{2}$
18. The sequence is geometric. $a_1 = \frac{10}{3}, r = \frac{2}{3}$
19. The sequence is geometric. $a_1 = 1, r = 2$
20. The sequence is geometric. $a_1 = -1, r = 1.06$
21. The sequence is not geometric.
22. The sequence is not geometric.
23. The sequence is geometric. $a_1 = \frac{1}{3}, r = \frac{1}{3}$
24. The sequence is geometric. $a_1 = 500, r = 10$
25. The sequence is geometric. $a_1 = \sqrt{5}, r = \sqrt{5}$
26. The sequence is not geometric.
27. $a_1 = 2, r = 5, a_n = 2 \cdot 5^{n-1}$
28. $a_1 = -3, r = 2, a_n = -3 \cdot 2^{n-1}$
29. $a_1 = 5, r = \frac{2}{3}, a_n = 5\left(\frac{2}{3}\right)^{n-1}$
30. $a_1 = 1, r = \sqrt{3}, a_n = \sqrt{3}^{n-1} = 3^{(n-1)/2}$
31. $a_1 = 0.2, r = -3, a_n = 0.2(-3)^{n-1}$
32. $a_1 = 1.3, r = -0.2, a_n = 1.3(-0.2)^{n-1}$
33. $a_1 = \pi^4, r = \pi^2, a_n = \pi^4(\pi^2)^{n-1} = \pi^{2n+2}$
34. $a_1 = e^2, r = \frac{1}{e^2}, a_n = e^2\left(\frac{1}{e^2}\right)^{n-1} = \frac{1}{e^{2n-4}}$
35. $a_7 = a_1 r^{7-1} = 5(2^6) = 320$
36. $a_7 = a_1 r^{7-1} = 8(3^6) = 5832$
37. $a_{10} = a_1 r^{10-1} = 3(-2)^9 = -1536$
38. $a_{10} = a_1 r^{10-1} = 7(-2)^9 = -3584$
39. $a_6 = a_1 r^{6-1} = \frac{1}{16}(3)^5 = \frac{243}{16}$
40. $a_6 = a_1 r^{6-1} = \frac{1}{81}(3)^5 = 3$
41. $a_9 = a_1 r^{9-1} = -1\left(\frac{5}{2}\right)^8 = -\frac{390,625}{256}$
42. $a_9 = a_1 r^{9-1} = -4\left(\frac{3}{4}\right)^8 = -\frac{6561}{16,384}$
43. $a_{20} = a_1 r^{20-1} = 500\left(-\frac{1}{2}\right)^{19} = -\frac{125}{131,072}$

44. $a_{20} = a_1 r^{20-1} = 1000\left(-\frac{1}{10}\right)^{19} = -\frac{1}{10^{16}}$

45. $a_1 = 5;\ r = \frac{10}{5} = 2$

$a_n = a_1 r^{n-1} \Rightarrow 5 \cdot 2^{n-1} = 5120 \Rightarrow$

$2^{n-1} = 1024 \Rightarrow 2^{n-1} = 2^{10} \Rightarrow$

$n - 1 = 10 \Rightarrow n = 11$

There are 11 terms in the sequence.

46. $a_1 = 2;\ r = \frac{6}{2} = 3$

$a_n = a_1 r^{n-1} \Rightarrow 2 \cdot 3^{n-1} = 4374 \Rightarrow$

$3^{n-1} = 2187 \Rightarrow 3^{n-1} = 3^7 \Rightarrow$

$n - 1 = 7 \Rightarrow n = 8$

There are 8 terms in the sequence.

47. $a_1 = 16;\ r = \frac{8}{16} = \frac{1}{2}$

$a_n = a_1 r^{n-1} \Rightarrow 16 \cdot \left(\frac{1}{2}\right)^{n-1} = \frac{1}{32} \Rightarrow$

$\left(\frac{1}{2}\right)^{n-1} = \frac{1}{16 \cdot 32} \Rightarrow \left(\frac{1}{2}\right)^{n-1} = \frac{1}{2^4 \cdot 2^5} = \frac{1}{2^9} \Rightarrow$

$n - 1 = 9 \Rightarrow n = 10$

There are 10 terms in the sequence.

48. $a_1 = 27;\ r = \frac{9}{27} = \frac{1}{3}$

$a_n = a_1 r^{n-1} \Rightarrow 27 \cdot \left(\frac{1}{3}\right)^{n-1} = \frac{1}{27} \Rightarrow$

$\left(\frac{1}{3}\right)^{n-1} = \frac{1}{27 \cdot 27} \Rightarrow \left(\frac{1}{3}\right)^{n-1} = \frac{1}{3^3 \cdot 3^3} = \frac{1}{3^6} \Rightarrow$

$n - 1 = 6 \Rightarrow n = 7$

There are 7 terms in the sequence.

49. $a_1 = 18;\ r = \frac{-12}{18} = -\frac{2}{3}$

$a_n = a_1 r^{n-1} \Rightarrow 18 \cdot \left(-\frac{2}{3}\right)^{n-1} = \frac{512}{729} \Rightarrow$

$\left(-\frac{2}{3}\right)^{n-1} = \frac{512}{729 \cdot 18} \Rightarrow$

$\left(-\frac{2}{3}\right)^{n-1} = \frac{2^9}{3^6 \cdot 3^2 \cdot 2} = \frac{2^8}{3^8} = \left(\frac{2}{3}\right)^8 = \left(-\frac{2}{3}\right)^8 \Rightarrow$

$n - 1 = 8 \Rightarrow n = 9$

There are 9 terms in the sequence.

50. $a_1 = 25;\ r = \frac{-15}{25} = -\frac{3}{5}$

$a_n = a_1 r^{n-1} \Rightarrow 25 \cdot \left(-\frac{3}{5}\right)^{n-1} = \frac{729}{625} \Rightarrow$

$\left(-\frac{3}{5}\right)^{n-1} = \frac{729}{625 \cdot 25} \Rightarrow$

$\left(-\frac{3}{5}\right)^{n-1} = \frac{3^6}{5^4 \cdot 5^2} = \frac{3^6}{5^6} = \left(\frac{3}{5}\right)^6 = \left(-\frac{3}{5}\right)^6 \Rightarrow$

$n - 1 = 6 \Rightarrow n = 7$

There are 7 terms in the sequence.

51. $r = 5, S_{10} = \frac{(1/10)(1 - 5^{10})}{1 - 5} = \frac{1{,}220{,}703}{5}$

52. $r = \frac{1}{3}, S_{10} = \frac{6(1 - (1/3)^{10})}{1 - (1/3)} = \frac{59{,}048}{6561}$

53. $r = -5, S_{12} = \frac{(1/25)(1 - 5^{12})}{1 - (-5)} = -\frac{40{,}690{,}104}{25}$

54. $r = -\frac{1}{100}, S_{12} = \frac{-10(1 - (-1/100)^{12})}{1 - (-1/100)}$

$= \frac{-10^{24} + 1}{(101)(10^{21})} = -\frac{10^{24} - 1}{(101)(10^{21})}$

55. $r = \frac{1}{4}, S_8 = \frac{5(1 - (1/4)^8)}{1 - 1/4} = \frac{109{,}225}{16{,}384}$

56. $r = \frac{1}{5}, S_8 = \frac{2(1 - (1/5)^8)}{1 - (1/5)} = \frac{195{,}312}{78{,}125}$

57. $\sum_{i=1}^{5} \left(\frac{1}{2}\right)^{i-1} = \frac{(1)(1 - (1/2)^5)}{1 - 1/2} = \frac{31}{16}$

58. $\sum_{i=1}^{5} \left(\frac{1}{5}\right)^{i-1} = \frac{(1)(1 - (1/5)^5)}{1 - 1/5} = \frac{781}{625}$

59. $a_1 = 3, r = \frac{2}{3}$

$\sum_{i=1}^{8} 3\left(\frac{2}{3}\right)^{i-1} = \frac{3(1 - (2/3)^8)}{1 - (2/3)} = \frac{6305}{729}$

60. $a_1 = 2, r = \frac{3}{5}$

$\sum_{i=1}^{8} 2\left(\frac{3}{5}\right)^{i-1} = \frac{(2)(1 - (3/5)^8)}{1 - (3/5)} = \frac{384{,}064}{78{,}125}$

61. $a_1 = \dfrac{1}{4}, r = 2$

$$\sum_{i=3}^{10} \frac{2^{i-1}}{4} = \sum_{i=1}^{10} \frac{2^{i-1}}{4} - \sum_{i=1}^{2} \frac{2^{i-1}}{4}$$
$$= \frac{(1/4)(1-2^{10})}{1-2} - \frac{(1/4)(1-2^2)}{1-2}$$
$$= \frac{1023}{4} - \frac{3}{4} = 255$$

62. $a_1 = 1/2, r = 5$

$$\sum_{i=3}^{10} \frac{5^{i-1}}{2} = \sum_{i=1}^{10} \frac{5^{i-1}}{2} - \sum_{i=1}^{2} \frac{5^{i-1}}{2}$$
$$= \frac{(1/2)(1-5^{10})}{1-5} - \frac{(1/2)(1-5^2)}{1-5}$$
$$= 1,220,703 - 3 = 1,220,700$$

63. $a_1 = -\dfrac{3}{5}, r = -\dfrac{5}{2}$

$$\sum_{i=1}^{20} \left(-\frac{3}{5}\right)\left(-\frac{5}{2}\right)^{i-1} = \frac{(-3/5)(1-(-5/2)^{20})}{1-(-5/2)}$$
$$= \frac{3(5^{20}-2^{20})}{35(2^{19})}$$

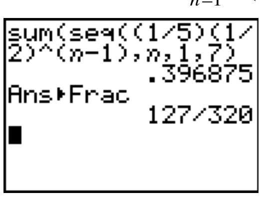
64. $a_1 = \left(-\dfrac{1}{4}\right)(3^{2-1}) = -\dfrac{3}{4}, r = \dfrac{1}{3}$

$$\sum_{i=1}^{20} \left(-\frac{1}{4}\right)(3^{2-i}) = \frac{-(3/4)(1-(1/3)^{20})}{1-(1/3)}$$
$$= \frac{-3^{20}+1}{8(3^{18})} = -\frac{3^{20}-1}{8(3^{18})}$$

65. $a_1 = \dfrac{1}{5};\ r = \dfrac{1}{2}$

$$\frac{1}{320} = \frac{1}{5}\left(\frac{1}{2}\right)^{n-1} \Rightarrow \frac{1}{64} = \left(\frac{1}{2}\right)^{n-1} \Rightarrow$$
$$\left(\frac{1}{2}\right)^{6} = \left(\frac{1}{2}\right)^{n-1} \Rightarrow 6 = n-1 \Rightarrow n = 7$$

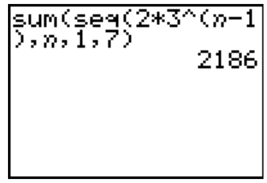
Thus, the sum is $\displaystyle\sum_{n=1}^{7} \frac{1}{5}\left(\frac{1}{2}\right)^{n-1}$.



66. $a_1 = 2;\ r = 3$

$1458 = 2 \cdot 3^{n-1} \Rightarrow 729 = 3^{n-1} \Rightarrow$
$3^6 = 3^{n-1} \Rightarrow 6 = n-1 \Rightarrow n = 7$

Thus, the sum is $\displaystyle\sum_{n=1}^{7} 2 \cdot 3^{n-1}$.



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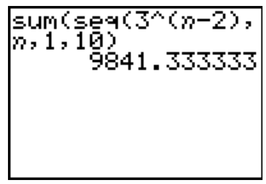

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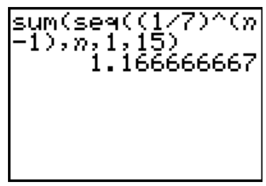

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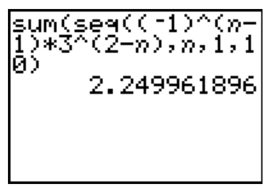

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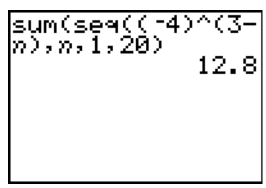

71. $a_1 = \dfrac{1}{3}, r = \dfrac{1}{3}$

$$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots = \frac{1/3}{1-(1/3)} = \frac{1}{2}$$

72. $a_1 = \dfrac{5}{2}, r = \dfrac{1}{2}$

$$\frac{5}{2} + \frac{5}{4} + \frac{5}{8} + \frac{5}{16} + \cdots = \frac{5/2}{1-(1/2)} = 5$$

73. $a_1 = -\dfrac{1}{2}, r = -\dfrac{1}{2}$

$$-\frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} + \cdots = \frac{-1/2}{1-(-1/2)} = -\frac{1}{3}$$

74. $a_1 = -\frac{3}{2}, r = -\frac{1}{2}$

$$-\frac{3}{2}+\frac{3}{4}-\frac{3}{8}+\frac{3}{16}+\cdots=\frac{-3/2}{1-(-1/2)}=-1$$

75. $a_1 = 8, r = -\frac{1}{4}$

$$8-2+\frac{1}{2}-\frac{1}{8}+\cdots=\frac{8}{1-(-1/4)}=\frac{32}{5}$$

76. $a_1 = 1, r = -\frac{3}{5}$

$$1-\frac{3}{5}+\frac{9}{25}-\frac{27}{125}+\cdots=\frac{1}{1-(-3/5)}=\frac{5}{8}$$

77. $a_1 = 5, r = \frac{1}{3}; \sum_{n=0}^{\infty} 5\left(\frac{1}{3}\right)^n = \frac{5}{1-(1/3)} = \frac{15}{2}$

78. $a_1 = 3, r = \frac{1}{4}; \sum_{n=0}^{\infty} 3\left(\frac{1}{4}\right)^n = \frac{3}{1-(1/4)} = 4$

79. $a_1 = 1, r = -\frac{1}{4}; \sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n = \frac{1}{1-(-1/4)} = \frac{4}{5}$

80. $a_1 = 1, r = -\frac{1}{3}; \sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n = \frac{1}{1-(-1/3)} = \frac{3}{4}$

11.3 Applying the Concepts

81. $a_5 = 20,000(1.03^5) = 23,185$

82. $a_{12} = 25(2^{12}) = 102,400$

83. $$a_{36} = 100\left(\frac{\left(1+\frac{0.06}{12}\right)^{36}-1}{\frac{0.06}{12}}\right) = \$3933.61$$

84. $$a_{20} = 300\left(\frac{\left(1+\frac{0.08}{2}\right)^{20}-1}{\frac{0.08}{2}}\right) = \$8933.42$$

85. $a_{10} = 2^{10} = 1024$.
A person has 1024 ancestors in the tenth generation back.

86. a. The first day there are 1000 bacteria, the second day, there are 2000 bacteria, and the third day, there are 4000 bacteria. Continuing in this manner, we find that $a_7 = 64,000 = 1000\left(2^{7-1}\right)$.

b. After n days, there will be $a_n = 1000\left(2^{n-1}\right)$ bacteria.

87. $a_1 = \$36,000, r = 1.05$

$$S_{20} = \frac{36,000(1-1.05^{20})}{1-1.05} = \$1,190,374.35$$

88. For option A, at \$4000 per month for 25 months, the buyer pays \$100,000. For option B, the buyer pays

$$1¢+2¢+4¢+8¢+\cdots=\frac{1(1-2^{25})}{1-2}$$

$= 33,554,431¢ = \$335,544.31$, so option A is the better choice.

89. $a_1 = 56.25, r = 0.8$

$$S_5 = \frac{56.25(1-0.8^5)}{1-0.8} = 189.09 \text{ cm}$$

90. $a_5 = 120,000(0.8^5) = \$39,321.60$

91. $$\begin{aligned} D &= 5+2(5)\left(\frac{3}{5}\right)+2(5)\left(\frac{3}{5}\right)^2+2(5)\left(\frac{3}{5}\right)^3+\ldots \\ &= 5+10\left(\frac{3}{5}\right)+10\left(\frac{3}{5}\right)^2+10\left(\frac{3}{5}\right)^3+\ldots \\ &= 5+\frac{6}{1-(3/5)} = 20 \text{ m} \end{aligned}$$

92. $$\begin{aligned} D &= 9+2(9)\left(\frac{3}{5}\right)+2(9)\left(\frac{3}{5}\right)^2++2(9)\left(\frac{3}{5}\right)^3\ldots \\ &= 9+18\left(\frac{3}{5}\right)+18\left(\frac{3}{5}\right)^2+18\left(\frac{3}{5}\right)^3+\ldots \\ &= 9+\frac{18(3/5)}{1-(3/5)} = 36 \text{ m} \end{aligned}$$

11.3 Beyond the Basics

93. The area of each square is 1/2 the area of the next larger square. However, we are including the shaded regions only, so $r = -1/2$. (This will eliminate the white squares.)

$$\sum_{i=1}^{\infty}\left(-\frac{1}{2}\right)^{i-1} = \frac{1}{1-(-1/2)} = \frac{2}{3}.$$

94. The perimeter of each triangle is 1/2 the perimeter of the next larger triangle. The perimeter of the largest triangle is 12, so the sum is $\sum_{n=1}^{\infty} 12\left(\frac{1}{2}\right)^n = \frac{12}{1-(1/2)} = 24$

95. Since $a_1, a_2, a_3, \ldots$ is a geometric sequence, $a_n = a_{n-1}r$ for every $n \geq 1$, where r is the common ratio. Taking the logarithm of both sides, we have $\ln a_n = \ln(a_{n-1}r)$ $= \ln a_{n-1} + \ln r$. So the sequence $\ln a_1, \ln a_2, \ln a_3, \ldots$ is arithmetic with common difference $\ln r$.

96. Since $a_1, a_2, a_3, \ldots$ is a geometric sequence, $a_n = a_{n-1}r$ for every $n \geq 1$, where r is the common ratio. Squaring both sides, we have $(a_n)^2 = (a_{n-1}r)^2 = (a_{n-1})^2 r^2$. So the sequence ${a_1}^2, {a_2}^2, {a_3}^2, \ldots$ is geometric with common ratio r^2.

97. Since $a_1, a_2, a_3, \ldots$ is a geometric sequence, $a_n = a_{n-1}r$ for every $n \geq 1$, where r is the common ratio. Taking the reciprocal of both sides, we have $\frac{1}{a_n} = \frac{1}{a_{n-1}r} = \frac{1}{a_{n-1}} \bullet \frac{1}{r}$. So the sequence $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \cdots$ is geometric with common ratio $1/r$.

98. Since $a_n = \frac{a_{n-1}}{c}$ for every $n \geq 1$, the sequence is geometric with common ratio $1/c$.

99. Since $a_n = \frac{2a_{n-1}}{x}$ for every $n \geq 1$, the sequence is geometric with common ratio $2/x$.

100. Since $a_n = -\frac{a_{n-1}}{x}$ for every $n \geq 1$, the sequence is geometric with common ratio $-a/x$.

101. Since $a_n = -\frac{a_{n-1}}{y}$ for every $n \geq 1$, the sequence is geometric with common ratio $-1/y$.

102. Since $a_1, a_2, a_3, \ldots$ is an arithmetic sequence, $a_n = a_{n-1} + d$ for every $n \geq 1$.

$a_n = a_{n-1} + d \Rightarrow 2^{a_n} = 2^{a_{n-1}+d} = 2^{a_{n-1}} \bullet 2^d$.

So the sequence $2^{a_1}, 2^{a_2}, 2^{a_3}, \ldots$ is geometric with common ratio 2^d.

11.4 Critical Thinking/Discussion/Writing

103. $a_{1001} = 10 + 1000(2.7) = 2710$.

$b_{1001} = 10(2^{1000})$, so b_{1001} is larger.

104. $\frac{a_1\left(1-\left(\frac{1}{2}\right)^{10}\right)}{1-\frac{1}{2}} = 1023 \Rightarrow a_1 = 512$

105. $5 + 5(2) + 5(2^2) + \cdots + 5(2^{15}) = \sum_{i=0}^{15} 5(2^i)$ or $\sum_{i=1}^{16} 5(2^{i-1})$.

106. Let $\frac{a}{r}, a,$ and ar be the three terms of the geometric sequence. Then we have

$$\begin{cases} \frac{a}{r} + a + ar = 35 \\ \left(\frac{a}{r}\right)a(ar) = 1000 \end{cases} \Rightarrow \begin{cases} a + ar + ar^2 = 35r \\ a^3 = 1000 \end{cases} \Rightarrow$$

$a = 10$

$10 + 10r + 10r^2 = 35r \Rightarrow 2r^2 - 5r + 2 = 0 \Rightarrow$

$(2r-1)(r-2) \Rightarrow r = \frac{1}{2}$ or $r = 2$.

If $r = \frac{1}{2}$, then the three numbers are 20, 10, and 5. If $r = 2$, then the numbers are also 5, 10, and 20.

107. Let $a, a+d,$ and $a+2d$ be the three terms of the arithmetic sequence. Then

$a + (a+d) + (a+2d) = 3a + 3d = 15 \Rightarrow$

$a + d = 5$.

The geometric sequence is $a+1, a+d+4$, and $a+2d+19$, and

$\frac{a+d+4}{a+1} = \frac{a+2d+19}{a+d+4}$.

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(*continued*)

Use substitution to solve the system:

$$\begin{cases} a+d=5 \\ \dfrac{a+d+4}{a+1}=\dfrac{a+2d+19}{a+d+4} \Rightarrow \end{cases}$$

$$\begin{cases} d=5-a \\ \dfrac{a+d+4}{a+1}=\dfrac{a+2d+19}{a+d+4} \Rightarrow \end{cases}$$

$$\frac{a+(5-a)+4}{a+1}=\frac{a+2(5-a)+19}{a+(5-a)+4} \Rightarrow$$

$$\frac{9}{a+1}=\frac{29-a}{9} \Rightarrow 29+28a-a^2=81 \Rightarrow$$

$$-(a^2-28a+52)=0 \Rightarrow (a-2)(a-26)=0 \Rightarrow$$

$a=2$ or $a=26$.

If $a = 2$, then $d = 3$, and the arithmetic sequence is 2, 5, 8. If $a = 26$, then $d = -21$, and the sequence is 26, 5, −16.

108. Let a, ar, and ar^2 be the three terms of the geometric sequence. So,

$a \cdot ar \cdot ar^2 = a^3r^3 = 1000 \Rightarrow ar = 10.$

The arithmetic sequence is a, $ar+6$, and ar^2+7. Then,

$(ar+6)-a=(ar^2+7)-(ar+6).$

Substituting $ar = 10$ gives

$(10+6)-a=(10r+7)-(10+6) \Rightarrow$

$16-a=10r-9 \Rightarrow 25=10r+a$

$ar=10 \Rightarrow r=\dfrac{10}{a}$

Substituting, we have

$25=10\left(\dfrac{10}{a}\right)+a \Rightarrow 25a=100+a^2 \Rightarrow$

$a^2-25a+100=0 \Rightarrow (a-5)(a-20)=0 \Rightarrow$

$a=5,\ a=20$

If $a = 5$, then we have $5r=10 \Rightarrow r=2,$ and the geometric sequence is 5, 10, 20.

If $a = 20$, then we have $20r=10 \Rightarrow r=\dfrac{1}{2},$ and the geometric sequence is 20, 10, 5.

11.3 Maintaining Skills

109. $P_n : n(n+1)$ is even.
$P_3 : 3(3+1)$ is even. True
$P_4 : 4(4+1)$ is even. True

110. $P_n : n^3+n$ is divisible by 3.
$P_3 : 3^3+3$ is divisible by 3. True
$P_4 : 4^3+4$ is divisible by 3. False

111. $P_n : 1+2+3+\cdots+n=\dfrac{n(n+1)}{2}$
$P_3 : 1+2+3=\dfrac{3(3+1)}{2}$ True
$P_4 : 1+2+3+4=\dfrac{4(4+1)}{2}$ True

112. $P_n : 2^n > 3n$
$P_3 : 2^3 > 3\cdot 3$ False
$P_4 : 2^4 > 3\cdot 4$ True

113. $P_n : n^2+n$ is divisible by 2.
$P_{n+1} : (n+1)^3+(n+1)$ is divisible by 2.

114. $P_n : 2^{3n}-1$ is divisible by 7.
$P_{n+1} : 2^{3(n+1)}-1$ is divisible by 7.

115. $P_n : 3^n > 5n$
$P_{n+1} : 3^{n+1} > 5(n+1)$

116. $P_n : 1+4+7+\cdots+(3n-2)=\dfrac{1}{2}n(3n-1)$

$$P_{n+1} : 1+4+7+\cdots+(3n-2)+(3(n+1)-2) = \frac{1}{2}(n+1)(3(n+1)-1) \Rightarrow$$

$$P_{n+1} : 1+4+7+\cdots+(3n-2)+(3n+1) = \frac{1}{2}(n+1)(3n+2)$$

117.
$$\frac{k(k+1)}{2}+(k+1)=\frac{k(k+1)}{2}+\frac{2(k+1)}{2} = \frac{k^2+k+2k+2}{2} = \frac{k^2+3k+2}{2} = \frac{(k+1)(k+2)}{2}$$

118.
$$\frac{1}{6}k(k+1)(2k+1)+(k+1)^2$$
$$=\frac{1}{6}(2k^3+3k^2+k)+(k^2+2k+1)$$
$$=\frac{1}{6}\left[(2k^3+3k^2+k)+6(k^2+2k+1)\right]$$
$$=\frac{1}{6}(2k^3+9k^2+13k+6)$$
$$=\frac{1}{6}(k+1)(k+2)(2k+3)$$

11.4 Mathematical Induction

11.4 Practice Problems

1. $P_{k+1}: \left[(k+1)+3\right]^2 > (k+1)^2 + 9 \Rightarrow$
$(k+4)^2 > (k+1)^2 + 9$

2. For $n = 1: 1 = \frac{1(1+1)}{2}$ is true. Assume that it is true for $n = k: 1+2+3+...+k = \frac{k(k+1)}{2}$.
Then for $n = k+1$: $1+2+3+...+k+(k+1)$
$= \frac{k(k+1)}{2} + (k+1) = \frac{k^2+k+2k+2}{2}$
$= \frac{(k+1)(k+2)}{2}$, which is exactly the statement for $n = k+1$. Therefore the formula is true for all natural numbers.

3. For $n = 1: 3^1 > 1$ is true. Assume that the inequality is true for $n = k: 3^k > k$. Then, we must use this fact to prove that for
$n = k+1: 3^{k+1} > k+1$.
$3^k > k \Rightarrow 3^k \cdot 3 > 3 \cdot k \Rightarrow 3^{k+1} > 3k > k+1$,
which is exactly the statement for $n = k+1$. Therefore the formula is true for all natural numbers.

11.4 Basic Concepts and Skills

1. Mathematical induction can only be used to prove statements about the natural numbers.

2. The first step in a mathematical induction proof that a statement P_n is true for all natural numbers *n* is that P_1 is true.

3. The second step in a mathematical induction proof is to assume that P_k is true for a natural number *k*, and then show that P_{k+1} is true.

4. True

5. False. The number *e* is a constant. The statement $e^n \geq n$ can be proven by mathematical induction for all natural numbers *n*.

6. True

7. $P_{k+1}: ((k+1)+1)^2 - 2(k+1)$
$= k^2 + 4k + 4 - 2k - 2$
$= k^2 + 2k + 2 = (k+1)^2 + 1$

8. $P_{k+1}: (1+(k+1))(1-(k+1)) = (k+2)(-k)$
$= -k^2 - 2k = -(k^2+2k) = 1-(k+1)^2$

9. $P_{k+1}: 2^{k+1} > 5(k+1)$

10. $P_{k+1}: 1+3+5+...+(2k-1)+(2k+1) = (k+1)^2$

11. For $n = 1: 2 = 2$ is true. Assume that it is true for $n = k$:
$2+4+6+...+2k = k^2 + k = k(k+1)$
Then for $n = k+1$:
$2+4+6+...+2k+2(k+1)$
$= k(k+1) + 2(k+1) = k^2 + 3k + 2$
$= (k+1)(k+2)$
which is exactly the statement for $n = k+1$. Therefore the formula is true for all natural numbers.

12. For $n = 1: 1 = 1^2$ is true. Assume that it is true for $n = k$: $1+3+5+...+(2k-1) = k^2$.
Then for $n = k+1$:
$1+3+5+...+(2k-1)+(2k+1)$
$= k^2 + (2k+1)$
$= (k+1)^2$
which is exactly the statement for $n = k+1$. Therefore the formula is true for all natural numbers.

13. For $n = 1: 4 = 2(1)(1+1)$ is true. Assume that it is true for $n = k$:
$4+8+12+...+4k = 2k(k+1)$
Then for $n = k+1$:
$4+8+12+...+4k+(4k+4)$
$= 2k(k+1)+(4k+4) = 2k^2+2k+4k+4$
$= 2(k^2+3k+2) = 2(k+1)(k+2)$
which is exactly the statement for $n = k+1$. Therefore the formula is true for all natural numbers.

14. For $n = 1: 3 = \dfrac{3(1)(1+1)}{2}$ is true. Assume that it is true for $n = k$:

$$3 + 6 + 9 + \ldots + 3k = \frac{3k(k+1)}{2}$$

Then for $n = k + 1$:

$$3 + 6 + 9 + \ldots + 3k + (3k+3)$$
$$= \frac{3k(k+1)}{2} + (3k+3)$$
$$= \frac{3k(k+1) + 2(3)(k+1)}{2} = \frac{3k^2 + 9k + 6}{2}$$
$$= \frac{3(k+1)(k+2)}{2}$$

which is exactly the statement for $n = k + 1$. Therefore the formula is true for all natural numbers.

15. For $n = 1: 1 = 1(2-1)$ is true. Assume that it is true for $n = k$:

$$1 + 5 + 9 + \ldots + (4k-3) = k(2k-1)$$

Then for $n = k + 1$:

$$1 + 5 + 9 + \ldots + (4k-3) + (4k+1)$$
$$= k(2k-1) + (4k+1) = 2k^2 - k + 4k + 1$$
$$= 2k^2 + 3k + 1 = (k+1)(2k+1)$$
$$= (k+1)(2(k+1)-1)$$

which is exactly the statement for $n = k + 1$. Therefore the formula is true for all natural numbers.

16. For $n = 1: 3 = \dfrac{1(6)}{2}$ is true. Assume that it is true for $n = k$:

$$3 + 8 + 13 + \ldots + (5k-2) = \frac{k(5k+1)}{2}$$

Then for $n = k + 1$:

$$3 + 8 + 13 + \ldots + (5k-2) + (5k+3)$$
$$= \frac{k(5k+1)}{2} + (5k+3) = \frac{5k^2 + k + 2(5k+3)}{2}$$
$$= \frac{5k^2 + 11k + 6}{2} = \frac{(k+1)(5k+6)}{2}$$
$$= \frac{(k+1)(5(k+1)+1)}{2}$$

which is exactly the statement for $n = k + 1$. Therefore the formula is true for all natural numbers.

17. For $n = 1: 3 = \dfrac{3(3-1)}{2}$ is true. Assume that it is true for $n = k$:

$$3 + 9 + 27 + \ldots + 3^k = \frac{3(3^k - 1)}{2}$$

Then for $n = k + 1$:

$$3 + 9 + 27 + \ldots + 3^k + 3^{k+1} = \frac{3(3^k - 1)}{2} + 3^{k+1}$$
$$= \frac{3^{k+1} - 3 + 2(3^{k+1})}{2}$$
$$= \frac{3(3^{k+1}) - 3}{2}$$
$$= \frac{3(3^{k+1} - 1)}{2}$$

which is exactly the statement for $n = k + 1$. Therefore the formula is true for all natural numbers.

18. For $n = 1: 5 = \dfrac{5(5^1 - 1)}{4}$ is true. Assume that it is true for $n = k$:

$$5 + 25 + 125 + \ldots + 5^k = \frac{5(5^k - 1)}{4}$$

Then for $n = k + 1$:

$$5 + 25 + 125 + \ldots + 5^k + 5^{k+1}$$
$$= \frac{5(5^k - 1)}{4} + 5^{k+1} = \frac{5^{k+1} - 5 + 4(5^{k+1})}{4}$$
$$= \frac{5(5^{k+1}) - 5}{4} = \frac{5(5^{k+1} - 1)}{4}$$

which is exactly the statement for $n = k + 1$. Therefore the formula is true for all natural numbers.

19. For $n = 1: \dfrac{1}{(1)(2)} = \dfrac{1}{1+1}$ is true. Assume that it is true for $n = k$:

$$\frac{1}{(1)(2)} + \frac{1}{(2)(3)} + \frac{1}{(3)(4)} + \ldots + \frac{1}{k(k+1)} = \frac{k}{k+1}.$$

Then for $n = k + 1$:

$$\frac{1}{(1)(2)} + \frac{1}{(2)(3)} + \frac{1}{(3)(4)} + \cdots$$
$$+ \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$$
$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k^2 + 2k + 1}{(k+1)(k+2)}$$

(*continued on next page*)

(*continued*)

$$= \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

which is exactly the statement for $n = k+1$. Therefore the formula is true for all natural numbers.

20. For $n = 1: \frac{1}{(2)(4)} = \frac{1}{4(1+1)}$ is true. Assume that it is true for $n = k$:

$$\frac{1}{(2)(4)} + \frac{1}{(4)(6)} + \frac{1}{(6)(8)} + ... + \frac{1}{2k(2k+2)} = \frac{k}{4(k+1)}$$

Then for $n = k+1$:

$$\frac{1}{(2)(4)} + \frac{1}{(4)(6)} + \frac{1}{(6)(8)} + \cdots + \frac{1}{2k(2k+2)} + \frac{1}{(2k+2)(2k+4)}$$
$$= \frac{k}{4(k+1)} + \frac{1}{(2k+2)(2k+4)},$$
$$= \frac{k^2+2k+1}{4(k+1)(k+2)}$$
$$= \frac{(k+1)^2}{4(k+1)(k+2)} = \frac{k+1}{4(k+2)},$$

which is exactly the statement for $n = k+1$. Therefore the formula is true for all natural numbers.

21. For $n = 1: 2 \le 2$ is true. Assume that it is true for $n = k$: $2 \le 2^k$. Then for $n = k+1$:

$2 \le 2^k \Rightarrow 2 \cdot 2 \le 2(2^k) \Rightarrow 4 \le 2^{k+1} \Rightarrow 2 \le 2^{k+1}$

Therefore the formula is true for all natural numbers.

22. For $n = 1: 2(1)+1 \le 3^1$ is true. Assume that it is true for $n = k$: $2k+1 \le 3^k$. Then for $n = k+1$:

$2k+1 \le 3^k \Rightarrow 3(2k+1) \le 3^{k+1} \Rightarrow$

$6k+3 \le 3^{k+1}$.

Since $2(k+1)+1 = 2k+3 \le 6k+3$,

$2(k+1)+1 \le 3^{k+1}$, which is exactly the statement for $n = k+1$. Therefore the formula is true for all natural numbers.

23. For $n = 1: 1(1+2) < (1+1)^2$ is true. Assume that it is true for $n = k$: $k(k+2) < (k+1)^2$. Then for $n = k+1$:

$(k+1)(k+1+2) = (k+1)(k+3) < (k+1+1)^2 \Rightarrow$

$k^2+4k+3 < k^2+4k+4 \Rightarrow$

$k^2+2k+2k+3 < k^2+2k+1+2k+3 \Rightarrow$

$k(k+2)+(2k+3) < (k+1)^2+(2k+3) \Rightarrow$

$k(k+2) < (k+1)^2$, which is exactly the statement for $n = k+1$. Therefore the formula is true for all natural numbers.

24. For $n = 1: 1 \le 1^2$ is true. Assume that it is true for $n = k$: $k \le k^2$. Then for $n = k+1$:

$k+1 \le (k+1)^2 \Rightarrow k+1 \le k^2+2k+1 \Rightarrow$

$-k \le k^2$, which is true for all natural numbers. Therefore the formula is true for all natural numbers.

25. For $n = 1: \frac{1!}{1} = (1-1)!$ is true. Assume that it is true for $n = k$: $\frac{k!}{k} = (k-1)!$. Then for $n = k+1$: $\frac{(k+1)!}{k+1} = \frac{(k+1)k!}{k+1} = k!$, which is exactly the statement for $n = k+1$. Therefore the formula is true for all natural numbers.

26. For $n = 1: \frac{1!}{(1+1)!} = \frac{1}{1+1}$ is true. Assume that it is true for $n = k$: $\frac{k!}{(k+1)!} = \frac{1}{k+1}$. Then for $n = k+1$:

$$\frac{(k+1)!}{(k+2)!} = \frac{(k+1)k!}{(k+2)(k+1)!} = \frac{k+1}{k+2}\left(\frac{k!}{(k+1)!}\right)$$
$$= \frac{k+1}{k+2}\left(\frac{1}{k+1}\right) = \frac{1}{k+2}$$

which is exactly the statement for $n = k+1$. Therefore the formula is true for all natural numbers.

27. For $n=1: 1^2=\dfrac{1(1+1)(2+1)}{6}$ is true. Assume that it is true for $n=k$:

$$1^2+2^2+3^2+\cdots+k^2=\frac{k(k+1)(2k+1)}{6}.$$

Then for $n=k+1$:

$$\begin{aligned}
&1^2+2^2+3^2+\ldots+k^2+(k+1)^2\\
&=\frac{k(k+1)(2k+1)}{6}+(k+1)^2\\
&=\frac{k(k+1)(2k+1)+6(k+1)^2}{6}\\
&=\frac{(k+1)\big(k(2k+1)+6(k+1)\big)}{6}\\
&=\frac{(k+1)(2k^2+7k+6)}{6}\\
&=\frac{(k+1)(k+2)(2k+3)}{6}
\end{aligned}$$

which is exactly the statement for $n=k+1$. Therefore the formula is true for all natural numbers.

28. For $n=1: 1^3=\dfrac{1^2(1+1)^2}{4}$ is true. Assume that it is true for $n=k$:

$$1^3+2^3+3^3+\ldots+k^3=\frac{k^2(k+1)^2}{4}$$

Then for $n=k+1$:

$$\begin{aligned}
&1^3+2^3+3^3+\ldots+k^3+(k+1)^3\\
&=\frac{k^2(k+1)^2}{4}+(k+1)^3\\
&=\frac{k^2(k+1)^2+4(k+1)^3}{4}\\
&=\frac{(k+1)^2\left(k^2+4(k+1)\right)}{4}\\
&=\frac{(k+1)^2(k^2+4k+4)}{4}=\frac{(k+1)^2(k+2)^2}{4}
\end{aligned}$$

which is exactly the statement for $n=k+1$. Therefore the formula is true for all natural numbers.

29. For $n=1$: 2 is a factor of $1^2+1=2$.
For $n=k$: assume that 2 is a factor of $n^2+n \Rightarrow k^2+k=2p$ for some integer p.
Then for $n=k+1$:

$$\begin{aligned}
(k+1)^2+(k+1)&=k^2+3k+2\\
&=(k^2+k)+(2k+2)\\
&=2p+2(k+1)=2(p+k+1)
\end{aligned}$$

$(p+k+1)$ is an integer, so $2(p+k+1)$ is divisible by 2. Therefore the formula is true for all natural numbers.

30. For $n=1$: 2 is a factor of $1^3+(5)(1)=6$. For $n=k$: Assume that 2 is a factor of $n^3+5n \Rightarrow k^3+5n=2p$ for some integer p.
Then for $n=k+1$:

$$\begin{aligned}
&(k+1)^3+5(k+1)\\
&\quad=k^3+3k^2+3k+1+5k+5\\
&\quad=k^3+5k+3k^2+3k+6=2p+6+3k(k+1)\\
&\quad=2(p+3)+3k(k+1)
\end{aligned}$$

Either k or $k+1$ is even, so $2(p+3)+3k(k+1)$ is divisible by 2. Therefore the formula is true for all natural numbers.

31. For $n=1: 6$ is a factor of $1(1+1)(1+2)=6$.
For $n=k$: Assume that 6 is a factor of $n(n+1)(n+2) \Rightarrow k(k+1)(k+2)=6p$ for some integer p.
Then for $n=k+1$:

$$\begin{aligned}
&(k+1)(k+2)(k+3)\\
&\quad=k(k+1)(k+2)+3(k+1)(k+2)\\
&\quad=6p+3(k+1)(k+2).
\end{aligned}$$

Either $k+1$ or $k+2$ is even, so $3(k+1)(k+2)$ is divisible by 6 and, thus, $6p+3(k+1)(k+2)$ is divisible by 6.
Therefore the formula is true for all natural numbers.

32. For $n=1$: 3 is a factor of $1(1+1)(1-1)=0$.
For $n=k$, assume that 3 is a factor of $n(n+1)(n-1) \Rightarrow k(k+1)(k-1)=3p$ for some integer p. Then for $n=k+1$:

$$\begin{aligned}
(k+1)(k+2)(k)&=k(k+1)(k-1+3)\\
&=k(k+1)(k-1)+3k(k+1)\\
&=3p+3k(k+1)\\
&=3(p+k(k+1))
\end{aligned}$$

Since $(p+k(k+1))$ is an integer, $3(p+k(k+1))$ is divisible by 3. Therefore the formula is true for all natural numbers.

33. For $n = 1: (ab)^1 = a^1b^1$ is true. Assume that it is true for $n = k$: $(ab)^k = a^kb^k$.
Then for $n = k+1$:
$(ab)^{k+1} = (ab)^k(ab) = a^kb^kab = a^{k+1}b^{k+1}$,
which is exactly the statement for $n = k+1$. Therefore the formula is true for all natural numbers.

34. For $n = 1: \left(\frac{a}{b}\right)^1 = \frac{a^1}{b^1}$ is true. Assume that it is true for $n = k$: $\left(\frac{a}{b}\right)^k = \frac{a^k}{b^k}$. Then for $n = k+1$:
$\left(\frac{a}{b}\right)^{k+1} = \left(\frac{a}{b}\right)^k\left(\frac{a}{b}\right) = \left(\frac{a^k}{b^k}\right)\left(\frac{a}{b}\right) = \frac{a^{k+1}}{b^{k+1}}$,
which is exactly the statement for $n = k+1$. Therefore the formula is true for all natural numbers.

11.4 Applying the Concepts

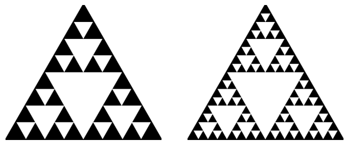
35. For $n = 2: \frac{2^2 - 2}{2} = 1$, which is the correct number of hugs for two people. Assume that the number of hugs for k people is $\frac{k^2 - k}{2}$. If there are $k + 1$ people, then we can separate one of them. The remaining k have $\frac{k^2 - k}{2}$ hugs by the hypothesis. The $k+1$ st person has to hug each of the other k people. So the number of hugs is
$\frac{k^2 - k}{2} + k = \frac{k^2 - k + 2k}{2} = \frac{k^2 + 2k + 1 - (k+1)}{2}$
$= \frac{(k+1)^2 - (k+1)}{2}$, which is exactly the statement for $n = k+1$. Therefore the formula is true for all natural numbers $n \geq 2$.

36. If there is one prize (i.e., $n = 1$), there are two ($= 2^1$) possible outcomes: either winning zero prizes or winning one prize. Assume that if there are k prizes, then there are 2^k possible outcomes. If there are $k + 1$ prizes, we can separate the possible outcomes into two categories: the ones containing the $k + 1$st prize and the ones not containing it. The number of possible outcomes not containing the $k + 1$st prize is 2^k by the hypothesis. We can obtain the number of outcomes including the $k + 1$st prize by including that prize with the possible outcomes that don't include it. Therefore, there are 2^k possible outcomes that include the $k + 1$st prize. So there are $2^k + 2^k = 2(2^k) = 2^{k+1}$ possible outcomes, which is exactly the formula for $n = k+1$. Therefore the formula is true for all natural numbers.

37. a. The number of sides of the nth figure is $3\left(4^{n-1}\right)$. For $n = 1: 3\left(4^0\right) = 3$, which is the number of sides of the triangle. Assume the number of sides of the kth figure is $3\left(4^{k-1}\right)$. When making the $k + 1$st figure, each new small triangle splits the original side into two sides, thus doubling the number of sides. In addition, each new triangle adds on two new sides per original side. So, the number of sides of the $k + 1$st figure is four times the number of sides of the kth figure, i.e., $4(3)(4^{k-1}) = 3(4^k)$, which is exactly the formula for $n = k+1$. Therefore the formula is true for all natural numbers.

b. The perimeter of the nth figure is $3\left(\frac{4}{3}\right)^{n-1}$.
For $n = 1: 3\left(\frac{4}{3}\right)^0 = 3$, which is the perimeter of the equilateral triangle with side length 1. Assume that the perimeter of the kth figure is $3\left(\frac{4}{3}\right)^{k-1}$. When making the $k + 1$st figure, each new small triangle adds on two new sides and deletes one small side, each with length $1/3$ of the original side. So, each new triangle increases the perimeter by $1/3$ of an original side. Therefore the perimeter of the $k + 1$st figure is $4/3$ times that of the kth figure, i.e., $\frac{4}{3}(3)\left(\frac{4}{3}\right)^{k-1} = 3\left(\frac{4}{3}\right)^k$, which is exactly the formula for $n = k+1$. Therefore the formula is true for all natural numbers.

38. a. There are 27 small black triangles + 9 small white triangles + 4 larger white triangles = 40 triangles.



b. There are 81 small black triangles + 27 small white triangles + 13 larger white triangles = 121 triangles.

c. The number of number of triangles after the *n*th iteration is $\frac{3^n-1}{2}$, and the number of the smallest triangles (i.e., the ones which were created during the last iteration) is 3^{n-1}. Prove as follows: For $n=1: \frac{3^1-1}{2}=1$ is the number of triangles after the first iteration, and the number of smallest triangles is $3^{1-1}=1$. Assume that the statement is true for $n=k$. When performing the *k* + 1st iteration, there is one new triangle on each side of the previous smallest triangles. That is $3(3^{k-1})=3^k$ new triangles. So, altogether there are $\frac{3^k-1}{2}+3^k = \frac{3^k-1+2(3^k)}{2}$ $=\frac{3(3^k)-1}{2}=\frac{3^{k+1}-1}{2}$, which is exactly the formula for $n=k+1$. Therefore the formula is true for all natural numbers.

39. The smallest number of moves to accomplish the transfer is 2^n-1. Prove as follows: for $n=1: 2^1-1=1$ is the number of necessary moves for one ring. Assume that the smallest number of moves for *k* rings is 2^k-1. When we move the biggest peg from the bottom to another peg, all the other rings must be stacked on the third peg in decreasing order from bottom to top. This requires 2^k-1 moves by the hypothesis. It takes one move to put the biggest ring onto the other peg. Then we have to move all the other rings on top of it, with again requires 2^k-1 moves. Altogether, then, there are $(2^k-1)+1+(2^k-1)=2(2^k)-1=2^{k+1}-1$ moves, which is exactly the formula for $n=k+1$. Therefore the formula is true for all natural numbers.

40. a. If n = 1, then two answer sheets are possible. If n = 2, then four answer sheets are possible.

b. The number of answer sheets is 2^n. Prove as follows: For $n=1: 2^1=2$, which is the number of answer sheets for one question. Assume that the number of answer sheets for *k* questions is 2^k. If there are *k* + 1 questions, then the first *k* can be answered in 2^k different ways, by the hypothesis. The last question can be answered in two ways. So the number of possible answer sheets is $2\left(2^k\right)=2^{k+1}$, which is exactly the formula for $n=k+1$. Therefore the formula is true for all natural numbers.

11.4 Beyond the Basics

41. For $n=1$, 5 is a factor of $8^1-3^1=5$. For $n=k$, assume that 5 is a factor of $8^n-3^n \Rightarrow$ $8^k-3^k=5p$ for some integer *p*. Then for $n=k+1$: $8^{k+1}-3^{k+1}=8(8^k)-3(3^k)$ $=8(8^k)-8(3^k)+5(3^k)=8(8^k-3^k)+5(3^k)$ $8(5p)+5(3^k)=5(8p+3^k)$. Since $8p+3^k$ is an integer, $5(8p+3^k)$ is divisible by 5. Therefore the formula is true for all natural numbers.

42. For $n=1$, 24 is a factor of $5^{2(1)}-1=24$. For $n=k$, assume that 24 is a factor of $5^{2n}-1 \Rightarrow 5^{2k}-1=24p$ for some integer *p*. Then for $n=k+1$: $5^{2(k+1)}-1=5^{2k+2}-1=5^2(5^{2k})-1$ $=25(5^{2k})-25+24=25(5^{2k}-1)+24$ $=25(24p)+24=24(25p+1)$. Since $25p+1$ is an integer, $24(25p+1)$ is divisible by 24. Therefore the formula is true for all natural numbers.

43. For $n = 1$, 64 is a factor of $3^{2(1)+2} - 8(1) - 9 = 64$. For $n = k$, assume that 64 is a factor of $3^{2n+2} - 8n - 9 \Rightarrow$ $3^{2k+2} - 8k - 9 = 64p$ for some integer p.
Then for $n = k + 1$:

$$\begin{aligned}
&3^{2(k+1)+2} - 8(k+1) - 9\\
&\quad = 3^{2k+4} - 8(k+1) - 9\\
&\quad = 3^2(3^{2k+2}) - 8k - 8 - 9\\
&\quad = 9(3^{2k+2}) - 9(8k) - 9(9) + 8(8k) + 64\\
&\quad = 9(3^{2k+2} - 8k - 9) + 64(k+1)\\
&\quad = 9(64p) + 64(k+1) = 64(9p + k + 1)
\end{aligned}$$

Since $9p + k + 1$ is an integer, $64(9p + k + 1)$ is divisible by 64. Therefore the formula is true for all natural numbers.

44. For $n = 1$, 64 is a factor of $9^1 - 8(1) - 1 = 0$. For $n = k$, assume that 64 is a factor of $9^n - 8n - 1 \Rightarrow 9^k - 8k - 1 = 64p$ for some integer p. Then for $n = k + 1$:

$$\begin{aligned}
9^{k+1} - 8(k+1) - 1 &= 9(9^k) - 8k - 8 - 1\\
&= 9(9^k) - 9(8k) - 9 + 8(8k)\\
&= 9(9^k - 8k - 1) + 64k\\
&= 9(64p) + 64k\\
&= 64(9p + k)
\end{aligned}$$

Since $9p + k$ is an integer, $64(9p + k)$ is divisible by 64. Therefore the formula is true for all natural numbers.

45. For $n = 1$, 3 is a factor of $2^{2(1)+1} + 1 = 9$. For $n = k$, assume that 3 is a factor of $2^{2n+1} + 1 \Rightarrow 2^{2k+1} + 1 = 3p$ for some integer p. Then for $n = k + 1$:

$$\begin{aligned}
2^{2(k+1)+1} + 1 = 2^{2k+3} + 1 &= 2^2 2^{2k+1} + 1\\
&= 2^2 2^{2k+1} + 4 - 3\\
&= 4(2^{2k+1} + 1) - 3\\
&= 4(3p) - 3 = 3(4p - 1)
\end{aligned}$$

Since $4p - 1$ is an integer, $3(4p - 1)$ is divisible by 3. Therefore the formula is true for all natural numbers.

46. For $n = 1$, 5 is a factor of $2^{4(1)} - 1 = 15$. For $n = k$, assume that 5 is a factor of $2^{4n} - 1 \Rightarrow$ $2^{4k} - 1 = 5p$ for some integer p. Then for $n = k + 1$:

$$\begin{aligned}
2^{4(k+1)} - 1 = 2^{4k+4} - 1 &= 2^4 2^{4k} - 1\\
&= 2^4 2^{4k} - 16 + 15\\
&= 16(2^{k+1} - 1) + 15\\
&= 16(5p) + 15 = 5(16p + 3)
\end{aligned}$$

Since $16p + 3$ is an integer, $5(16p + 3)$ is divisible by 5. Therefore the formula is true for all natural numbers.

47. For $n = 1$, $a - b$ is a factor of $a^1 - b^1 = a - b$. For $n = k$, assume that $a - b$ is a factor of $a^n - b^n \Rightarrow a^k - b^k = (a - b)P(a,b)$ where $P(a,b)$ is a polynomial of a and b. Then for $n = k + 1$:

$$\begin{aligned}
a^{k+1} - b^{k+1} &= a\left(a^k - b^k\right) + b^k\left(a - b\right)\\
&= a\left(a - b\right)P(a,b) + b^k\left(a - b\right)\\
&= \left(a - b\right)\left(aP(a,b) + b^k\right)
\end{aligned}$$

Since $aP(a,b) + b^k$ is a polynomial of a and b, $a - b$ is a factor of $\left(a - b\right)\left(aP(a,b) + b^k\right)$. Therefore the statement is true for all natural numbers.

48. For $n = 1: 1 = \dfrac{a^1 - 1}{a - 1}$ is true. Assume that it is true for $n = k: 1 + a + a^2 + \ldots + a^{k-1} = \dfrac{a^k - 1}{a - 1}$. Then for $n = k + 1$:

$$\begin{aligned}
1 + a + a^2 + \ldots + a^{k-1} + a^k &= \frac{a^k - 1}{a - 1} + a^k\\
&= \frac{a^k - 1 + a^k(a - 1)}{a - 1}\\
&= \frac{a^{k+1} - 1}{a - 1}
\end{aligned}$$

which is exactly the formula for $n = k + 1$. Therefore the formula is true for all natural numbers.

49. For $n=1$: $\sum_{k=1}^{2}\frac{1}{k+1}=\frac{1}{2}+\frac{1}{3}\le\frac{5}{6}$ is true. Assume that it is true for $n=m$: $\sum_{k=1}^{m+1}\frac{1}{k+m}\le\frac{5}{6}$.

Then for $n=m+1$: $\sum_{k=1}^{m+2}\frac{1}{k+m+1}=\sum_{k=2}^{m+3}\frac{1}{k+m}=\left(\sum_{k=1}^{m+1}\frac{1}{k+m}\right)+\frac{1}{2m+2}+\frac{1}{2m+3}-\frac{1}{m+1}$.

In the last expression, $\frac{1}{2m+2}+\frac{1}{2m+3}-\frac{1}{m+1}=\frac{(2m+3)+(2m+2)-2(2m+3)}{2(m+1)(2m+3)}=-\frac{1}{2(m+1)(2m+3)}<0$

So, $\sum_{k=1}^{m+2}\frac{1}{k+m+1}=\left(\sum_{k=1}^{m+1}\frac{1}{k+m}\right)+\frac{1}{2m+2}+\frac{1}{2m+3}-\frac{1}{m+1}<\left(\sum_{k=1}^{m+1}\frac{1}{k+m}\right)\le\frac{5}{6}$

This is exactly the formula for $n=k+1$. Therefore the formula is true for all natural numbers.

11.4 Critical Thinking/Discussion/Writing

50. The proof is not valid because the first step of the mathematical induction, which is to show that the statement is true for P_1, is missing from this proof.

51. For $n=m=4$: $2^4\ge 4^2\Rightarrow 16\ge 16$ is true. Now assume that the statement is true for $n=k$: $2^k\ge k^2$. Now we must use P_k to prove that P_{k+1} is true. That is, $P_{k+1}: 2^{k+1}\ge (k+1)^2$.

$2\cdot 2^k\ge 2k^2\Rightarrow 2^{k+1}\ge 2k^2\Rightarrow 2^{k+1}\ge k^2+k^2$

Since $k\ge 4$, we have $2k\ge 8\Rightarrow 2k>1$. Also, since $k\ge 4$, we have $k^2\ge 4k$.

$2^{k+1}>k^2+k^2>k^2+4k\Rightarrow 2^{k+1}>k^2+4k\Rightarrow 2^{k+1}>k^2+2k+2k\Rightarrow$

$2^{k+1}>k^2+2k+2k>k^2+2k+1\Rightarrow 2^{k+1}>k^2+2k+1\Rightarrow 2^{k+1}>(k+1)^2$

Thus, $P_k\Rightarrow P_{k+1}$, and the statement is true for all natural numbers.

52. For $n=m=6$, $6!>6^3\Rightarrow 720>216$ is true. Now assume that the statement is true for $n=k$: $k!>k^3$. Now we must use P_k to prove that P_{k+1} is true. That is, $P_{k+1}:(k+1)!\ge(k+1)^3$.

$k!>k^3\Rightarrow (k+1)\cdot k!\ge(k+1)\cdot k^3\Rightarrow (k+1)!\ge(k+1)\cdot k^3$.

Since $k+1\ge 7$, we have $(k+1)!\ge(k+1)\cdot k^3\Rightarrow(k+1)!\ge 7k^3$.

We need to show that $7k^3\ge(k+1)^3$ for $k\ge 6$:

$(k+1)^3=k^3+3k^2+3k+1\Rightarrow 7k^3\ge k^3+3k^2+3k+1\Rightarrow 6k^3\ge 3k^2+3k+1\Rightarrow$

$k^3\ge\frac{1}{2}k^2+\frac{1}{2}k+\frac{1}{6}\Rightarrow k^3\ge\frac{1}{2}\left(k^2+k+\frac{1}{3}\right)\Rightarrow 7k^3\ge\frac{7}{2}\left(k^2+k+\frac{1}{3}\right)$

$k\ge 6\Rightarrow 7k^2\cdot k\ge 7k^2\cdot 6\Rightarrow 7k^3\ge 42k^2$

Now compare $\frac{7}{2}\left(k^2+k+\frac{1}{3}\right)$ with $42k^2$:

$12\left[\frac{7}{2}\left(k^2+k+\frac{1}{3}\right)\right]=42k^2+42k+14\Rightarrow 42k^2\ge\frac{7}{2}\left(k^2+k+\frac{1}{3}\right)$, so

$7k^3\ge\frac{7}{2}\left(k^2+k+\frac{1}{3}\right)\Rightarrow(k+1)!\ge\frac{7}{2}\left(k^2+k+\frac{1}{3}\right)\Rightarrow(k+1)!\ge(k+1)^3$.

11.4 Maintaining Skills

53. $(x+y)^2=(x+y)(x+y)=x^2+2xy+y^2$

54. $(x+y)^3=(x+y)(x+y)^2=(x+y)\left(x^2+2xy+y^2\right)=x^3+2x^2y+xy^2+yx^2+2xy^2+y^3$
$=x^3+3x^2y+3xy^2+y^3$

55. $(x+y)^4=(x+y)(x+y)^3=(x+y)\left(x^3+3x^2y+3xy^2+y^3\right)$
$=x^4+3x^3y+3x^2y^2+xy^3+yx^3+3x^2y^2+3xy^3+y^4=x^4+4x^3y+6x^2y^2+4xy^3+y^4$

56. $(x+y)^5=(x+y)(x+y)^4=(x+y)\left(x^4+4x^3y+6x^2y^2+4xy^3+y^4\right)$
$=x^5+4x^4y+6x^3y^2+4x^2y^3+xy^4+yx^4+4x^3y^2+6x^2y^3+4xy^4+y^5$
$=x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$

57. In exercise 53, we see that the sum of the exponents in each term is 2. In exercise 54, the sum of the exponents in each term is 3. In exercise 55, the sum of the exponents in each term is 4, and in exercise 56, the sum is 5. Thus, we can say that the sum of the exponents on x and y in each term in the expansion of $(x+y)^n$ is n.

58. In exercise 53, we see that there are 3 terms. In exercise 54, there are 4 terms. In exercise 55, there are 5 terms, and in exercise 56, there are 6 terms. Thus, we can say that there are $n + 1$ terms in the expansion of $(x+y)^n$.

11.5 The Binomial Theorem

11.5 Practice Problems

1. $(3y-x)^6=1(3y)^6+6(3y)^5(-x)+15(3y)^4(-x)^2+20(3y)^3(-x)^3+15(3y)^2(-x)^4+6(3y)(-x)^5+1(-x)^6$
$=729y^6-1458y^5x+1215y^4x^2-540y^3x^3+135y^2x^4-18yx^5+x^6$

2. a. $\binom{6}{2}=\dfrac{6!}{2!(6-2)!}=\dfrac{6\cdot5\cdot4!}{2!4!}=15$

b. $\binom{12}{9}=\dfrac{12!}{9!(12-9)!}=\dfrac{12\cdot11\cdot10\cdot9!}{9!3!}=\dfrac{12\cdot11\cdot10}{3\cdot2\cdot1}=220$

3. $(3x-y)^4=\binom{4}{0}(3x)^4+\binom{4}{1}(3x)^3(-y)+\binom{4}{2}(3x)^2(-y)^2+\binom{4}{3}(3x)(-y)^3+\binom{4}{4}(-y)^4$
$=\dfrac{4!}{0!4!}\left(81x^4\right)-\dfrac{4!}{1!3!}\left(27x^3y\right)+\dfrac{4!}{2!2!}\left(9x^2y^2\right)-\dfrac{4!}{3!1!}\left(3xy^3\right)+\dfrac{4!}{4!0!}y^4$
$=81x^4-108x^3y+54x^2y^2-12xy^3+y^4$

4. The term x^3y^9 is the tenth term in the expansion of $(x+y)^{12}$. So its coefficient is $\binom{12}{9}$.

$\binom{12}{9}=\dfrac{12!}{9!(12-9)!}=\dfrac{12\cdot11\cdot10\cdot9!}{9!3!}=\dfrac{12\cdot11\cdot10}{3\cdot2\cdot1}=220$

5. $\binom{n}{n-r}x^r(2a)^{n-r}=\binom{15}{15-3}x^3(2a)^{15-3}=\binom{15}{12}x^3(2a)^{12}=\dfrac{15!}{12!(15-12)!}x^3(2a)^{12}=\dfrac{15!}{12!3!}x^3(2a)^{12}$
$=\dfrac{15\cdot14\cdot13\cdot12!}{12!3!}x^3(2a)^{12}=\dfrac{15\cdot14\cdot13}{3\cdot2\cdot1}x^3a^{12}=455x^3(2a)^{12}=1,863,680x^3a^{12}$

11.5 Basic Concepts and Skills

1. The expansion of $(x+y)^n$ has <u>$n + 1$</u> terms.

2. In the expansion of $(x+y)^5$, the coefficient of x^2y^3 is $\binom{n}{r}$ when $n = \underline{5}$ and $r = \underline{3}$.

3. Expanding a difference, such as $(2x-y)^{10}$, results in <u>alternating signs</u> between terms.

4. For any positive integer n, $\binom{n}{n} = \underline{1}$.

5. False. If n is even, then there are an odd number of terms in the expansion, so one coefficient appears just once.

6. False. $2\binom{n}{2} = n!$ for $n > 3$.

7. $\frac{6!}{3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3!} = 120$

8. $\frac{11!}{9!} = \frac{11 \cdot 10 \cdot 9!}{9!} = 110$

9. $\frac{12!}{11!} = \frac{12 \cdot 11!}{11!} = 12$

10. $\frac{3!}{0!} = \frac{3 \cdot 2 \cdot 1}{1} = 6$

11. $\binom{6}{4} = \frac{6!}{4!(6-4)!} = \frac{6 \cdot 5 \cdot 4!}{4!2!} = 15$

12. $\binom{6}{3} = \frac{6!}{3!(6-3)!} = \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3!3!} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20$

13. $\binom{9}{0} = \frac{9!}{0!(9-0)!} = \frac{9!}{0!9!} = 1$

14. $\binom{12}{0} = \frac{12!}{0!(12-0)!} = \frac{12!}{0!12!} = 1$

15. $\binom{7}{1} = \frac{7!}{1!(7-1)!} = \frac{7 \cdot 6!}{1!6!} = 7$

16. $\binom{7}{3} = \frac{7!}{3!(7-3)!} = \frac{7 \cdot 6 \cdot 5 \cdot 4!}{3!4!} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$

17. $(x+2)^4 = x^4 + 4(2x^3) + 6(2^2x^2) + 4(2^3x) + 2^4 = x^4 + 8x^3 + 24x^2 + 32x + 16$

18. $(x+3)^4 = x^4 + 4(3x^3) + 6(3^2x^2) + 4(3^3x) + 3^4 = x^4 + 12x^3 + 54x^2 + 108x + 81$

19. $(x-2)^5 = x^5 + 5(-2)x^4 + 10(-2)^2x^3 + 10(-2)^3x^2 + 5(-2)^4x + (-2)^5 = x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$

20. $(3-x)^5 = 3^5 + 5(3^4)(-x) + (10)(3^3)(-x)^2 + (10)(3^2)(-x)^3 + (5)(3)(-x)^4 + (-x)^5$
$= 243 - 405x + 270x^2 - 90x^3 + 15x^4 - x^5$

21. $(2-3x)^3 = 2^3 + 3(2^2)(-3x) + 3(2)(-3x)^2 + (-3x)^3 = 8 - 36x + 54x^2 - 27x^3$

22. $(3-2x)^3 = 3^3 + 3(3^2)(-2x) + 3(3)(-2x)^2 + (-2x)^3 = 27 - 54x + 36x^2 - 8x^3$

23. $(2x+3y)^4 = (2x)^4 + 4(2x)^3(3y) + 6(2x)^2(3y)^2 + 4(2x)(3y)^3 + (3y)^4$
$= 16x^4 + 96x^3y + 216x^2y^2 + 216xy^3 + 81y^4$

24. $(2x+5y)^4 = (2x)^4 + 4(2x)^3(5y) + 6(2x)^2(5y)^2 + 4(2x)(5y)^3 + (5y)^4$
$= 16x^4 + 160x^3y + 600x^2y^2 + 1000xy^3 + 625y^4$

25. $(x+1)^4 = x^4 + 4(1x^3) + 6(1^2x^2) + 4(1^3x) + 1^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$

26. $(x+2)^4 = x^4 + 4(2x^3) + 6(2^2x^2) + 4(2^3x) + (2)^4 = x^4 + 8x^3 + 24x^2 + 32x + 16$

27. $(x-1)^5 = x^5 + 5(-1)x^4 + 10(-1)^2x^3 + 10(-1)^3x^2 + 5(-1)^4x + (-1)^5 = x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1$

28. $(1-x)^5 = 1^5 + 5(1^4)(-x) + (10)(1^3)(-x)^2 + (10)(1^2)(-x)^3 + (5)(1)(-x)^4 + (-x)^5$
$= 1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5$

29. $(y-3)^3 = y^3 + 3(-3)y^2 + 3(-3)^2y + (-3)^3 = y^3 - 9y^2 + 27y - 27$

30. $(2-y)^5 = 2^5 + 5(2^4)(-y) + (10)(2^3)(-y)^2 + (10)(2^2)(-y)^3 + (5)(2)(-y)^4 + (-y)^5$
$= 32 - 80y + 80y^2 - 40y^3 + 10y^4 - y^5$

31. $$(x+y)^6 = \binom{6}{0}x^6 + \binom{6}{1}x^5y + \binom{6}{2}x^4y^2 + \binom{6}{3}x^3y^3 + \binom{6}{4}x^2y^4 + \binom{6}{5}xy^5 + \binom{6}{6}y^6$$
$$= \frac{6!}{0!(6-0)!}x^6 + \frac{6!}{1!(6-1)!}x^5y + \frac{6!}{2!(6-2)!}x^4y^2 + \frac{6!}{3!(6-3)!}x^3y^3 + \frac{6!}{4!(6-4)!}x^2y^4 + \frac{6!}{5!(6-5)!}xy^5 + \frac{6!}{6!(6-6)!}y^6$$
$$= x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$$

32. $$(x-y)^6 = \binom{6}{0}x^6 + \binom{6}{1}x^5(-y) + \binom{6}{2}x^4(-y)^2 + \binom{6}{3}x^3(-y)^3 + \binom{6}{4}x^2(-y)^4 + \binom{6}{5}x(-y)^5 + \binom{6}{6}(-y)^6$$
$$= \frac{6!}{0!(6-0)!}x^6 - \frac{6!}{1!(6-1)!}x^5y + \frac{6!}{2!(6-2)!}x^4y^2 - \frac{6!}{3!(6-3)!}x^3y^3 + \frac{6!}{4!(6-4)!}x^2y^4 - \frac{6!}{5!(6-5)!}xy^5 + \frac{6!}{6!(6-6)!}y^6$$
$$= x^6 - 6x^5y + 15x^4y^2 - 20x^3y^3 + 15x^2y^4 - 6xy^5 + y^6$$

33. $(1+3y)^5 = 1^5 + 5(1^4)(3y) + (10)(1^3)(3y)^2 + (10)(1^2)(3y)^3 + (5)(1)(3y)^4 + (3y)^5$
$= 1 + 15y + 90y^2 + 270y^3 + 405y^4 + 243y^5$

34. $(2x+1)^5 = (2x)^5 + 5(2x)^4(1) + (10)(2x)^3(1)^2 + (10)(2x)^2(1)^3 + (5)(2x)(1)^4 + (1)^5$
$= 32x^5 + 80x^4 + 80x^3 + 40x^2 + 10x + 1$

35. $(2x+1)^4 = (2x)^4 + 4(2x)^3(1) + 6(2x)^2(1)^2 + 4(2x)(1)^3 + 1^4 = 16x^4 + 32x^3 + 24x^2 + 8x + 1$

36. $(3x-1)^4 = (3x)^4 + 4(3x)^3(-1) + 6(3x)^2(-1)^2 + 4(3x)(-1)^3 + (-1)^4 = 81x^4 - 108x^3 + 54x^2 - 12x + 1$

37. $(x-2y)^3 = x^3 + 3(x^2)(-2y) + 3(x)(-2y)^2 + (-2y)^3 = x^3 - 6x^2y + 12xy^2 - 8y^3$

38. $(2x-y)^3 = (2x)^3 + 3(2x)^2(-y) + 3(2x)(-y)^2 + (-y)^3 = 8x^3 - 12x^2y + 6xy^2 - y^3$

39. $(2x+y)^4 = (2x)^4 + 4(2x)^3y + 6(2x)^2y^2 + 4(2x)y^3 + y^4 = 16x^4 + 32x^3y + 24x^2y^2 + 8xy^3 + y^4$

40. $$\begin{aligned}(3x-2y)^4 &= (3x)^4 + 4(3x)^3(-2y) + 6(3x)^2(-2y)^2 + 4(3x)(-2y)^3 + (-2y)^4\\ &= 81x^4 - 216x^3y + 216x^2y^2 - 96xy^3 + 16y^4\end{aligned}$$

41. $$\begin{aligned}\left(\frac{x}{2}+2\right)^7 &= \binom{7}{0}\left(\frac{x}{2}\right)^7 + \binom{7}{1}\left(\frac{x}{2}\right)^6(2) + \binom{7}{2}\left(\frac{x}{2}\right)^5(2^2) + \binom{7}{3}\left(\frac{x}{2}\right)^4(2^3) + \binom{7}{4}\left(\frac{x}{2}\right)^3(2^4) + \binom{7}{5}\left(\frac{x}{2}\right)^2(2^5)\\ &\quad + \binom{7}{6}\left(\frac{x}{2}\right)(2^6) + \binom{7}{7}(2^7)\\ &= \frac{7!}{0!(7-0)!}\left(\frac{x}{2}\right)^7 + \frac{7!}{1!(7-1)!}\left(\frac{x}{2}\right)^6(2) + \frac{7!}{2!(7-2)!}\left(\frac{x}{2}\right)^5(2^2) + \frac{7!}{3!(7-3)!}\left(\frac{x}{2}\right)^4(2^3)\\ &\quad + \frac{7!}{4!(7-4)!}\left(\frac{x}{2}\right)^3(2^4) + \frac{7!}{5!(7-5)!}\left(\frac{x}{2}\right)^2(2^5) + \frac{7!}{6!(7-6)!}\left(\frac{x}{2}\right)(2^6) + \frac{7!}{7!(7-7)!}(2^7)\\ &= \frac{x^7}{128} + \frac{7x^6}{32} + \frac{21x^5}{8} + \frac{35x^4}{2} + 70x^3 + 168x^2 + 224x + 128\end{aligned}$$

42. $$\begin{aligned}\left(2-\frac{x}{2}\right)^7 &= \binom{7}{0}(2)^7 + \binom{7}{1}(2)^6\left(-\frac{x}{2}\right) + \binom{7}{2}(2)^5\left(-\frac{x}{2}\right)^2 + \binom{7}{3}(2^4)\left(-\frac{x}{2}\right)^3 + \binom{7}{4}(2^3)\left(-\frac{x}{2}\right)^4\\ &\quad + \binom{7}{5}(2^2)\left(-\frac{x}{2}\right)^5 + \binom{7}{6}(2)\left(-\frac{x}{2}\right)^6 + \binom{7}{7}\left(-\frac{x}{2}\right)^7\\ &= \frac{7!}{0!(7-0)!}(2)^7 + \frac{7!}{1!(7-1)!}(2)^6\left(-\frac{x}{2}\right) + \frac{7!}{2!(7-2)!}(2)^5\left(-\frac{x}{2}\right)^2 + \frac{7!}{3!(7-3)!}(2)^4\left(-\frac{x}{2}\right)^3\\ &\quad + \frac{7!}{4!(7-4)!}(2)^3\left(-\frac{x}{2}\right)^4 + \frac{7!}{5!(7-5)!}(2)^2\left(-\frac{x}{2}\right)^5 + \frac{7!}{6!(7-6)!}(2)\left(-\frac{x}{2}\right)^6 + \frac{7!}{7!(7-7)!}\left(-\frac{x}{2}\right)^7\\ &= -\frac{x^7}{128} + \frac{7x^6}{32} - \frac{21x^5}{8} + \frac{35x^4}{2} - 70x^3 + 168x^2 - 224x + 128\end{aligned}$$

43. $\left(a^2-\frac{1}{3}\right)^4 = (a^2)^4 + 4(a^2)^3\left(-\frac{1}{3}\right) + 6(a^2)^2\left(-\frac{1}{3}\right)^2 + 4(a^2)\left(-\frac{1}{3}\right)^3 + \left(-\frac{1}{3}\right)^4 = a^8 - \frac{4}{3}a^6 + \frac{2}{3}a^4 - \frac{4}{27}a^2 + \frac{1}{81}$

44. $\left(\frac{1}{2}-a^2\right)^4 = \left(\frac{1}{2}\right)^4 + 4\left(\frac{1}{2}\right)^3(-a^2) + 6\left(\frac{1}{2}\right)^2(-a^2)^2 + 4\left(\frac{1}{2}\right)(-a^2)^3 + (-a^2)^4 = a^8 - 2a^6 + \frac{3}{2}a^4 - \frac{1}{2}a^2 + \frac{1}{16}$

45. $\left(\frac{1}{x}+y\right)^3 = \left(\frac{1}{x}\right)^3 + 3\left(\frac{1}{x}\right)^2y + 3\left(\frac{1}{x}\right)y^2 + y^3 = \frac{1}{x^3} + \frac{3y}{x^2} + \frac{3y^2}{x} + y^3$

46. $\left(x+\frac{2}{y}\right)^3 = x^3 + 3\left(\frac{2}{y}\right)x^2 + 3\left(\frac{2}{y}\right)^2x + \left(\frac{2}{y}\right)^3 = x^3 + \frac{6x^2}{y} + \frac{12x}{y^2} + \frac{8}{y^3}$

47. The term containing x^7 is the fourth term in the expansion: $\binom{10}{7}x^7y^3 = \frac{10!}{7!(10-7)!}x^7y^3 = 120x^7y^3$

48. The term containing y^7 is the eighth term in the expansion: $\binom{10}{7}x^3y^7 = \frac{10!}{7!(10-7)!}x^3y^7 = 120x^3y^7$

49. The term containing x^3 is the tenth term in the expansion:

$$\binom{12}{3}x^3(-2)^9 = \frac{12!}{3!(12-3)!}(-512)x^3 = -112{,}640x^3$$

50. The term containing x^3 is the fourth term in the expansion:

$$\binom{12}{9}x^3(-2)^9 = \frac{12!}{9!(12-9)!}(-512)x^3 = -112{,}640x^3$$

51. The term containing x^6 is the third term in the expansion:

$$\binom{8}{2}(2x)^6(3y)^2 = \frac{8!}{2!(8-2)!}(64x^6)(9y^2) = 16{,}128x^6y^2$$

52. The term containing y^6 is the seventh term in the expansion:

$$\binom{8}{6}(2x)^2(3y)^6 = \frac{8!}{6!(8-6)!}\left(4x^2\right)\left(729y^6\right) = 81{,}648x^2y^6$$

53. The term containing y^9 is the tenth term in the expansion:

$$\binom{11}{9}(5x)^2(-2y)^9 = \frac{11!}{2!(11-2)!}\left(25x^2\right)\left(-512y^9\right) = -704{,}000x^2y^9$$

54. The term containing x^9 is the seventh term in the expansion:

$$\binom{15}{6}(7x)^9(-y)^6 = \frac{15!}{6!(15-6)!}\left(40{,}353{,}607x^9\right)\left(y^6\right) = 201{,}969{,}803{,}035x^9y^6$$

55. $(1.2)^5 = (1+0.2)^5 = 1^5 + 5(1)^4(0.2) + 10(1)^3(0.2)^2 + 10(1)^2(0.2)^3 + 5(1)(0.2)^4 + (0.2)^5 = 2.48832$

56. a. $(2.9)^4 = (2+0.9)^4 = 2^4 + 4(2^3)(0.9) + 6(2^2)(0.9)^2 + 4(2)(0.9)^3 + (0.9)^4 = 70.7281$

b. $(10.4)^3 = (10+0.4)^3 = (10^3) + 3(10^2)(0.4) + 3(10)(0.4)^2 + (0.4)^3 = 1124.864$

11.5 Beyond the Basics

57. The middle term is the sixth term in the expansion:

$$\binom{10}{5}\left(\sqrt{x}\right)^5\left(-\frac{2}{x^2}\right)^5 = \frac{10!}{5!(10-5)!}x^2\sqrt{x}\left(-\frac{32}{x^{10}}\right) = -\frac{8064\sqrt{x}}{x^8}$$

58. The middle term is the sixth term in the expansion:

$$\binom{10}{5}\left(\sqrt{x}\right)^5\left(\frac{3}{3x^2}\right)^5 = \frac{10!}{5!(10-5)!}x^2\sqrt{x}\left(\frac{3^5}{3^5x^{10}}\right) = \frac{252\sqrt{x}}{x^8}$$

59. The middle term is the seventh term in the expansion:

$$\binom{12}{6}(1^6)(-x^2y^{-3})^6 = \frac{12!}{6!(12-6)!}\left(\frac{x^{12}}{y^{18}}\right) = \frac{924x^{12}}{y^{18}}$$

60. The middle term is $\binom{2n}{n}(1^n)x^n = \frac{(2n)!}{n!n!}x^n = \frac{[(1)(3)(5)\cdots(2n-1)][(2)(4)(6)\cdots(2n)]}{n!n!}x^n$ (arranging the factors in groups of even factors and odd factors) $= \frac{[(1)(3)(5)\cdots(2n-1)][2(1)\cdot 2(2)\cdot 2(3)\cdot\cdots\cdot 2(n)]}{n!n!}x^n$ (there are n factors of 2; factor out 2 in the even factors) $= \frac{[(1)(3)(5)\cdots(2n-1)](2^n)n!}{n!n!}x^n = \frac{(1)(3)(5)\cdots(2n-1)}{n!}2^n x^n$

61. By the Binomial Theorem, we have

$$2^n = (1+1)^n = \binom{n}{0}(1)^n(1)^0 + \binom{n}{1}(1)^{n-1}(1)^1 + \binom{n}{2}(1)^{n-2}(1)^2 + \cdots + \binom{n}{n}(1)^0(1)^n$$
$$= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n}$$

62. By the Binomial Theorem, we have

$$0 = (1-1)^n = \binom{n}{0}(1)^n(1)^0 + \binom{n}{1}(1)^{n-1}(-1)^1 + \binom{n}{2}(1)^{n-2}(-1)^2 - \binom{n}{3}(1)^{n-3}(-1)^2 + \cdots + \binom{n}{n}(1)^0(-1)^n$$
$$= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n\binom{n}{n}$$

63. $$\binom{k}{j} + \binom{k}{j-1} = \frac{k!}{j!(k-j)!} + \frac{k!}{(j-1)!(k-j+1)!} = \frac{k!(j-1)!(k-j+1)! + k!j!(k-j)!}{j!(k-j)!(j-1)!(k-j+1)!}$$
$$= \frac{[k!(j-1)!(k-j)!][j+(k-j+1)]}{j!(k-j)!(j-1)!(k-j+1)!} = \frac{k!(j+(k-j+1))}{j!(k+1-j)!} = \frac{k!(k+1)}{j!(k+1-j)!} = \frac{(k+1)!}{j!(k+1-j)!} = \binom{k+1}{j}$$

64. For $n=1: (x+y)^1 = \binom{1}{0}x^1y^0 + \binom{1}{1}x^0y^1 = x+y$ is true. For $n=k$, assume that the hypothesis is true, i.e,

$(x+y)^k = \sum_{j=0}^{k}\binom{k}{j}x^{k-j}y^j$. Then for $n=k+1$,

$$(x+y)^{k+1} = (x+y)(x+y)^k = x(x+y)^k + y(x+y)^k$$
$$= x\sum_{j=0}^{k}\binom{k}{j}x^{k-j}y^j + y\sum_{j=0}^{k}\binom{k}{j}x^{k-j}y^j \quad \text{(By the hypothesis)}$$
$$= \sum_{j=0}^{k}\binom{k}{j}x^{k-j+1}y^j + \sum_{j=0}^{k}\binom{k}{j}x^{k-j}y^{j+1} \quad \text{(Multiply through by } x \text{ and } y.)$$
$$= x^{k+1} + \sum_{j=1}^{k}\binom{k}{j}x^{k-j+1}y^j + \sum_{j=0}^{k}\binom{k}{j}x^{k-j}y^{j+1} \quad \text{(Pull out the } j \text{ term.)}$$
$$= x^{k+1} + \sum_{j=1}^{k}\binom{k}{j}x^{k-j+1}y^j + \sum_{j=1}^{k}\binom{k}{j-1}x^{k-j+1}y^j \quad \text{(Pull out the } j=k+1 \text{ term.)}$$
$$= x^{k+1} + \sum_{j=1}^{k}\binom{k}{j}x^{k-j+1}y^j + \sum_{j=1}^{k}\binom{k}{j-1}x^{k-j+1}y^j + y^{k+1}$$
$$= (x^{k+1} + y^{k+1}) + \left(\sum_{j=1}^{k}\binom{k}{j}x^{k-j+1}y^j + \sum_{j=1}^{k}\binom{k}{j-1}x^{k-j+1}y^j\right)$$

(*continued on next page*)

(*continued*)

$$= (x^{k+1} + y^{k+1}) + \sum_{j=1}^{k}\left[\binom{k}{j} + \binom{k}{j-1}\right]\left(x^{k-j+1}y^j\right)$$

$$= (x^{k+1} + y^{k+1}) + \sum_{j=1}^{k}\binom{k+1}{j}\left(x^{k-j+1}y^j\right) = \sum_{j=0}^{k+1}\binom{k+1}{j}\left(x^{k-j+1}y^j\right)$$

This is exactly the formula for $n = k+1$. Therefore, the theorem is true for all natural numbers.

65. $(2x-1)^4 + 4(2x-1)^3(3-2x) + 6(2x-1)^2(3-2x)^2 + 4(2x-1)(3-2x)^3 + (3-2x)^4$
$= \left((2x-1)+(3-2x)\right)^4 = 2^4 = 16$

66. $(x+1)^4 - 4(x+1)^3(x-1) + 6(x+1)^2(x-1)^2 - 4(x+1)(x-1)^3 + (x-1)^4 = \left((x+1)-(x-1)\right)^4 = 2^4 = 16$

67. $(3x-1)^5 + 5(3x-1)^4(1-2x) + 10(3x-1)^3(1-2x)^2 + 10(3x-1)^2(1-2x)^3 + 5(3x-1)(1-2x)^4 + (1-2x)^5$
$= \left((3x-1)+(1-2x)\right)^5 = x^5$

68. a. The constant term is the seventh term in the expansion: $\binom{9}{6}(x^2)^3\left(-\frac{1}{x}\right)^6 = \frac{9!}{6!(9-6)!}\frac{x^6}{x^6} = 84$

b. The constant term is the third term in the expansion: $\binom{10}{2}\left(\sqrt{x}\right)^8\left(-\frac{2}{x^2}\right)^2 = \frac{10!}{2!(10-2)!}\left(\frac{4x^4}{x^4}\right) = 180$

69. The constant term is the third term in the expansion: $240 = \binom{6}{2}(kx)^4\left(-\frac{1}{x^2}\right)^2$

$$= \frac{6!}{2!(6-2)!}k^4x^4\left(\frac{1}{x^4}\right) = 15k^4 \Rightarrow k^4 = 16 \Rightarrow k = \pm 2$$

70. The constant term is the fourth term in the expansion: $1320 = \binom{11}{3}\left(x^3\right)^8\left(\frac{k}{x^8}\right)^3$

$$= \frac{11!}{3!(11-3)!}\left(x^{24}\right)\left(\frac{k^3}{x^{24}}\right) = 165k^3 \Rightarrow k^3 = 8 \Rightarrow k = 2$$

71. The general term in the expansion has the form $\binom{11}{k}(2x^2)^k\left(-\frac{1}{4x}\right)^{11-k} = \binom{11}{k}(2^k)\left(-\frac{1}{4}\right)^{11-k}\frac{x^{2k}}{x^{11-k}}$. In order to get a constant term, $2k$ must equal $11 - k$. However, the solution of this equation is not an integer, so there is no constant term in the expansion.

11.5 Critical Thinking/Discussion/Writing

72. $2^6 = (1+1)^6$

$$= \binom{6}{0}(1)^6(1)^0 + \binom{6}{1}(1)^5(1)^1 + \binom{6}{2}(1)^4(1)^2 + \binom{6}{3}(1)^3(1)^3 + \binom{6}{4}(1)^2(1)^4 + \binom{6}{5}(1)^1(1)^5 + \binom{6}{6}(1)^0(1)^6$$

$$= \binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6}$$

73. $0=(1-1)^{10}$

$$=\binom{10}{0}(1)^{10}(1)^0-\binom{10}{1}(1)^9(1)^1+\binom{10}{2}(1)^8(1)^2-\binom{10}{3}(1)^7(1)^3+\binom{10}{4}(1)^6(1)^4-\binom{10}{5}(1)^5(1)^5+\binom{10}{6}(1)^4(1)^6$$
$$-\binom{10}{7}(1)^3(1)^7+\binom{10}{8}(1)^2(1)^8-\binom{10}{9}(1)^1(1)^9+\binom{10}{10}(1)^9(1)^{10}$$
$$=\binom{10}{0}-\binom{10}{1}+\binom{10}{2}-\binom{10}{3}+\binom{10}{4}-\binom{10}{5}+\binom{10}{6}-\binom{10}{7}+\binom{10}{8}-\binom{10}{9}+\binom{10}{10}$$

74. $(x+y)^2=x^2+2x+y^2$. If $x\ge 0$ and $y\ge 0$, then $2xy\ge 0$ and
$x^2+2xy+y^2\ge x^2+y^2\Rightarrow$
$(x+y)^2\ge x^2+y^2\Rightarrow x+y\ge\sqrt{x^2+y^2}$

75. $(x+y)^n=x^n+\binom{n}{1}x^{n-1}y+\binom{n}{2}x^{n-2}y^2+\cdots$
$+\binom{n}{n-1}xy^{n-1}+y^n$. If $x>0$ and $y>0$, then all the intermediate terms in the expansion are positive. Therefore $(x+y)^n>x^n+y^n$ if $n>1$.

76. $(1+x)^n=1+\binom{n}{1}x+\binom{n}{2}x^2+...+\binom{n}{n}x^n$
$=1+nx+\binom{n}{1}x+\binom{n}{2}x^2+...+\binom{n}{n}x^n$. If $x>$ 0, then all the terms in the sum are positive, and
$1+nx+\binom{n}{1}x+\binom{n}{2}x^2+...+\binom{n}{n}x^n>1+nx$.
Therefore, $(1+x)^n>1+nx$ if $n>1$.

11.5 Maintaining Skills

77. $5\cdot 6\cdot 7\cdot 8=\dfrac{8!}{4!}$

78. $10\cdot 9\cdot 8\cdot 7\cdot 6=\dfrac{10!}{5!}$

79. $2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12$
$=2\cdot(2\cdot 2)\cdot(2\cdot 3)\cdot(2\cdot 4)\cdot(2\cdot 5)\cdot(2\cdot 6)$
$=2^6\cdot 6!$

80. $3\cdot 6\cdot 9\cdot 12\cdot 15=3\cdot(3\cdot 2)\cdot(3\cdot 3)\cdot(3\cdot 4)\cdot(3\cdot 5)$
$=3^5\cdot 5!$

81. $\dfrac{12!}{10!}=12\cdot 11=132$

82. $\dfrac{100!}{(100-2)!}=\dfrac{100!}{98!}=100\cdot 99=9900$

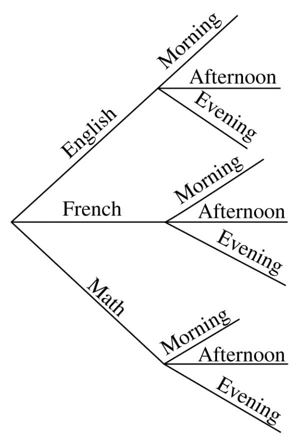
83. $\dfrac{8!}{5!3!}=\dfrac{8\cdot 7\cdot 6}{3\cdot 2\cdot 1}=56$

84. $\dfrac{10!}{(10-3)!3!}=\dfrac{10!}{7!3!}=\dfrac{10\cdot 9\cdot 8}{3\cdot 2\cdot 1}=120$

Section 11.6 Counting Principles

11.6 Practice Problems

1.



There are 9 ways to choose a course.

2. There are 7 restaurants and 5 movies, so there are $7\cdot 5=35$ different choices.

3. There are $10^6=1{,}000{,}000$ possibilities.

4. There are $7!=7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1=5040$ possible arrangements.

5. There are $9\cdot 8\cdot 7\cdot 6=3024$ possibilities.

6. a. $P(9,2)=\dfrac{9!}{(9-2)!}=\dfrac{9\cdot 8\cdot 7!}{7!}=72$

b. $P(n,0)=\dfrac{n!}{(n-0)!}=\dfrac{n!}{n!}=1$

7. {bears, bulls, lions}, {bears, bulls, tigers}, {bears, lions, tigers}, {bulls, lions, tigers}
$C(4, 3) = 4$

8. a. $C(9,2) = \dfrac{9!}{(9-2)!2!} = \dfrac{9 \cdot 8 \cdot 7!}{7!2!} = 36$

b. $C(n,0) = \dfrac{n!}{(n-0)!0!} = \dfrac{n!}{n!} = 1$

c. $C(n,n) = \dfrac{n!}{(n-n)!n!} = \dfrac{n!}{n!} = 1$

9. $C(12,3) = \dfrac{12!}{(12-3)!3!} = \dfrac{12 \cdot 11 \cdot 10 \cdot 9!}{9!3!}$
$= \dfrac{12 \cdot 11 \cdot 10}{3 \cdot 2 \cdot 1} = 220$
There are 220 ways that three of the twelve chocolates can be chosen.

10. $\dfrac{6!}{2!2!2!} = 90$
There are 90 ways to send six counselors in pairs to three different locations.

11.6 Basic Concepts and Skills

1. Any arrangement of n distinct objects in a fixed order in which no object is used more than once is called a permutation.

2. The number of permutations of 5 distinct objects is $5! = 120$.

3. When r objects are chosen from n distinct objects, the set of r objects is called a combination of n objects taken r at a time.

4. The number of distinguishable permutations of 10 objects of which 3 are of one kind and 7 are of a second kind is $\dfrac{10!}{3!7!} = 120$.

5. True

6. True (as long as $r > 1$)

7. $P(6,1) = \dfrac{6!}{(6-1)!} = 6$

8. $P(7,3) = \dfrac{7!}{(7-3)!} = 210$

9. $P(8,2) = \dfrac{8!}{(8-2)!} = 56$

10. $P(10,6) = \dfrac{10!}{(10-6)!} = 151,200$

11. $P(9,9) = \dfrac{9!}{(9-9)!} = 362,880$

12. $P(5,5) = \dfrac{5!}{(5-5)!} = 120$

13. $P(7,0) = \dfrac{7!}{(7-0!)} = 1$

14. $P(4,0) = \dfrac{4!}{(4-0)!} = 1$

15. $C(8,3) = \dfrac{8!}{(8-3)!3!} = 56$

16. $C(7,2) = \dfrac{7!}{(7-2)!2!} = 21$

17. $C(9,4) = \dfrac{9!}{(9-4)!(4!)} = 126$

18. $C(10,5) = \dfrac{10!}{(10-5)!5!} = 252$

19. $C(5,5) = \dfrac{5!}{(5-5)!(5!)} = 1$

20. $C(6,6) = \dfrac{6!}{(6-6)!6!} = 1$

21. $C(3,0) = \dfrac{3!}{(3-0)!0!} = 1$

22. $C(5,0) = \dfrac{5!}{(5-0)!0!} = 1$

23. a. The first letter can be chosen in 26 different ways. The second letter can also be chosen in 26 different ways, so the number of different 2-letter codes is $26 \cdot 26 = 676$.

b. The first letter can be chosen in 26 different ways. The second letter can be chosen in 25 different ways, so the number of different 2-letter codes is $26 \cdot 25 = 650$.

24. Each question can be answered in one of two ways. There are ten questions, so the number of possible answer sheets is $2^{10} = 1024$.

25. The president can be chosen in 50 ways, while the vice-president can be chosen in 49 ways. So the number of possibilities is $50 \cdot 49 = 2450$.

26. The first digit cannot be 0, so there are 9 possibilities for the first digit. There are 10 possibilities for each of the remaining three digits, so there are $9 \cdot 10 \cdot 10 \cdot 10 = 9000$ possible four digit numbers.

27. There are four people to be seated in a row of four chairs, so there are $P(4,4) = \frac{4!}{(4-4)!} = 24$ different ways to arrange the people.

28. There are ten teams and three possible outcomes, so there are $P(10,3) = \frac{10!}{(10-3)!}$ $= 720$ possible outcomes.

29. There are five shirts, three pairs of pants, and four ties, so there are $5 \cdot 3 \cdot 4 = 60$ possible outfits.

30. There are eight pieces in the program, so there are $P(8,8) = \frac{8!}{(8-8)!} = 40{,}320$ possible program arrangements.

31. There are five different toppings. So there are $C(5,2) = \frac{5!}{(5-2)!2!} = 10$ different pizzas with two additional toppings.

32. There are 15 students, so it is possible to choose two representatives in $C(15,2) = \frac{15!}{(15-2)!(2)!} = 105$ different ways.

33. There are eleven problems, so it is possible to choose nine of them in $C(11,9) = \frac{11!}{(11-9)!(9!)} = 55$ different ways.

34. There are 18 potential teammates, so it is possible to choose three of them in $C(18,3) = \frac{18!}{(18-3)!(3!)} = 816$ different ways.

35. Order is not important, so find the number of combinations. There are $C(11,4) = 330$ different combinations.

36. There are seven applicants for the clerk position and four applicants for the technician position, so there are $7 \cdot 4 = 28$ possibilities.

37. Order is important, so find the number of permutations. There are 8 seats and six people will be seated, so there are $P(8,6) = \frac{8!}{(8-6)!} = 20{,}160$ different ways to seat 6 people.

38. To rent the canoes, there are $C(7,2) = \frac{7!}{(7-2)!(2!)} = 21$ different combinations. There are $C(5,3) = \frac{5!}{(5-3)!(3)!} = 10$ different combinations to rent the kayaks. So there are $21 \cdot 10 = 210$ different ways to choose the students for the rentals.

39. Order is important, so find the number of permutations or use the Fundamental Counting Principle. There are $P(5,5) = \frac{5!}{(5-5)!} = 120$ ways to visit the five colleges.

40. Order is important, so find the number of permutations or use the Fundamental Counting Principle. There are $75 \cdot 74 \cdot 73 \cdot 72 = 29{,}170{,}800$ different ways to call the numbers.

11.6 Applying the Concepts

41. a. The first digit cannot be a zero or a one, so there are eight possibilities for the first digit. There are two possibilities for the second digit and ten possibilities for the third digit. So there are $8 \cdot 2 \cdot 10 = 160$ possible three-digit area codes.

b. There are eight possibilities for the first digit, ten possibilities for the second digit, and ten possibilities for the third digit. So there are $8 \cdot 10 \cdot 10 = 800$ possible three-digit area codes.

42. If the call letters begin with K, there are $26^2 = 676$ radio stations with three letters and $26^3 = 17{,}576$ radio stations with four letters, for a total of 18,252 possibilities. There are the same number of possibilities if the call letters begin with W, so there are a total of $2 \cdot 18{,}252 = 36{,}504$ different call-letter selections.

43. a. If no repetitions are allowed, there are $24 \cdot 23 \cdot 22 = 12{,}144$ three-letter fraternity names.

b. If repetitions are allowed, there are $24^3 =$ 13,824 three-letter fraternity names.

44. a. If no letters are repeated, there are $10 \cdot 9 \cdot 8 =$ 720 three-letter codes.

b. If letters can be repeated, there are $10^3 =$ 1000 three-letter codes.

45. To reach a majority decision, five, six, seven, eight or all nine of the nine justices must agree. Since order is not important, find the sum of the combinations

$$C(9,5)+C(9,6)+C(9,7)+C(9,8)+C(9,9)$$
$$=\frac{9!}{(9-5)!(5!)}+\frac{9!}{(9-6)!(6!)}+\frac{9!}{(9-7)!(7!)}+\frac{9!}{(9-8)!(8!)}+\frac{9!}{(9-9)!(9!)}$$
$$=126+84+36+9+1=256$$

There are 256 ways to form a majority of five from the nine justices.

46. Order is not important, so find the combination $C(10,4)=\frac{10!}{(10-4)!4!}=210$.

There are 210 different ways to select a committee of four from a group of ten.

47. Al can choose from $8 \cdot 8 \cdot 8 = 512$ different outfits. Since there are 365 days in a year, he can wear a different outfit every day.

48. There are $2 \cdot 3 \cdot 8 \cdot 5 = 240$ different dinners.

49. a. There are 7! = 5040 different ways to arrange the houses since any arrangement of the seven houses could be made to correspond to a permutation of the seven designs by agreeing that the first four go on one side of the street and the remaining three go on the other side of the street.

b. There are 7! = 5040 different ways to arrange the houses since any arrangement of the seven houses could be made to correspond to a permutation of the seven designs by agreeing that the first five go on one side of the street and the remaining two go on the other side of the street.

50. There are $P(7,7) = 5040$ possible arrangements. Only one is in increasing order of difficulty, so there are 5039 ways the author could fail to arrange the exercises correctly.

51. a. There are seven letters in the word CHARITY. If two letters in each four-letter group must be an A and an R, then there are five letters that can be chosen in groups of two: $C(5,2)=\frac{5!}{(5-2)!(2)!}=10$ possibilities.

b. If one letter in each four-letter group must be an A and none of the letters in the group can be an R, then there are five letters that can be chosen in groups of three: $C(5,3)=\frac{5!}{(5-3)!(3)!}=10$ possibilities.

c. If none of the letters in each four-letter group can be an A or an R, there there are five letters that can be chosen in groups of four: $C(5,4)=\frac{5!}{(5-4)!(4!)}=5$ possibilities.

52. There are $C(20,4)=\frac{20!}{(20-4)!(4!)}=4845$ possible combinations of four students and $C(8,2)=\frac{8!}{(8-2)!2!}=28$ possible combinations of two professors. So, there are $28 \cdot 4845 = 135{,}660$ possible combinations of four students and two professors.

53. The store will need to stock $5 \cdot 7 \cdot 3 = 105$ shirts to have one of each type.

54. Because the tenth digit is determined by the first nine, we need to count only the possible arrangements of the first nine digits. Each of the nine digits has ten possibilities, so there are $10^9 = 1{,}000{,}000{,}000$ possible ISBNs.

55. The maximum number of folders is necessary if all of the questionnaires have different rankings. There are $10! = 3{,}628{,}800$ possible rankings, so they would need that many folders.

56. Since order is important, find the permutation. There are $12! = 479{,}001{,}600$ different orders in which the parcels could be delivered.

57. In each group of four, if one child is "constant", then the other three children can be chosen in $C(6,3) = \frac{6!}{(6-3)!3!} = 20$ ways.

So the maximum number of times any one child will go to the circus is 20.

There are $C(7,4) = \frac{7!}{(7-4)!4!} = 35$ combinations of four children, so the father will go to the circus 35 times.

58. The person selects a total of nine books, three out of six history books, two out of four biology books, and four out of five economics books. There are $C(6,3) = \frac{6!}{(6-3)!3!} = 20$ possible combinations of history books, $C(4,2) = \frac{4!}{(4-2)!2!} = 6$ combinations of biology books, and $C(5,4) = \frac{5!}{(5-4)!4!} = 5$ combinations of economics books. Then there are $20 \times 6 \times 5 = 600$ ways to select the nine books. Each of these 600 appropriate selections of books can be arranged in 9! ways for a total of $600 \times 9! = 217{,}728{,}000$

59. Because order is not important, we use combinations. Under the "B", there are $C(15,5) = \frac{15!}{(15-5)!5!} = 3003$ combinations of numbers. There are the same number under the "I", "G", and "O". Under the "N", there are $C(15,4) = \frac{15!}{(15-4)!4!} = 1365$. So there are $3003^4 \cdot 1365 = 111{,}007{,}923{,}832{,}370{,}565$ different bingo cards.

60. Each team in the league plays 10 games against the teams in its division and 6 games against the teams in the other division. Because the game where team A plays against team B is the same as the game where team B plays against team A, there will be $16 \cdot 6 = 96$ games.

61. There are 10 people and four are needed for the committee. Once the first person is chosen, that person's spouse is eliminated, so there are 8 ways to choose the next person. That person's spouse cannot be chosen next, so there are 6 ways to choose the third person, and 4 ways to choose the fourth person. That gives 1920 possible permutations. However, the order that people are chosen for the committee is not important, so the number of combinations is $\frac{1920}{4!} = 80$.

62. The three-person committee must have at least one man. There are $C(4,3) = \frac{4!}{(4-3)!3!} = 4$ ways to choose the committee if it consists of all men. There are $C(4,2) \times C(7,1)$

$= \frac{4!}{(4-2)!2!} \times \frac{7!}{(7-1)!(1!)} = 6 \times 7 = 42$ ways to select two men and one woman. There are $C(4,1) \times C(7,2) = \frac{4!}{(4-1)!1!} \times \frac{7!}{(7-2)!(2!)}$

$= 4 \times 21 = 84$ ways to select one man and two women. So there are 4 + 42 + 84 = 130 ways to select a three-person committee that includes at least one man.

63. If Cory wants to give his sister one coin, there are 4 different combinations of one coin. If he wants to give his sister two coins, there are $C(4,2) = 6$ different combinations. If he wants to give his sister three coins, there are 4 combinations, and if he wants to give his sister four coins, there is one combination. So there are 4 + 6 + 4 + 1 = 15 different combinations of coins.

64. There are $2 \cdot 3 \cdot 5 \cdot 6 = 180$ different four-course meals.

65. Out of six letters, there are two "A"s and the rest are singles. So, there are $\frac{6!}{2!1!1!1!1!} = 360$ distinguishable ways to arrange the letters.

66. Out of seven letters, there are two "A"s and the rest are singles. So, there are $\frac{7!}{2!1!1!1!1!1!} = 2520$ distinguishable ways to arrange the letters.

67. Out of seven letters, there are three "S"s, two "C"s, and the rest are singles. So, there are $\frac{7!}{3!2!1!1!} = 420$ distinguishable ways to arrange the letters.

68. Out of seven letters, there are two "A"s, two "R"s, and the rest are singles. So, there are $\dfrac{7!}{2!2!1!1!1!} = 1260$ distinguishable ways to arrange the letters.

11.6 Beyond the Basics

69. The company needs n workers in combinations of three. So $C(n,3) = \dfrac{n!}{(n-3)!3!} \geq 20 \Rightarrow$

$\dfrac{n(n-1)(n-2)(n-3)!}{(n-3)!3!} \geq 20 \Rightarrow n^3 - 3n^2 + 2n \geq 120 \Rightarrow n^3 - 3n^2 + 2n - 120 \geq 0$

The possible rational zeros are $\pm1, \pm2, \pm3, \pm4,\ \pm5, \pm6, \pm8, \pm10, \pm12, \pm15, \pm20, \pm24, \pm30, \pm60,\ \pm120$.

Using synthetic division, we find that 6 is a zero:

$$\begin{array}{r|rrrr} 6 & 1 & -3 & 2 & -120 \\ & & 6 & 18 & 120 \\ \hline & 1 & 3 & 20 & 0 \end{array}$$

and $(n-6)(n^2 + 3n + 20) = 0$.

The two zeros of $n^2 + 3n + 20 = 0$ are complex, so we reject them. Therefore, six different workers are necessary.

70. The radio station needs n songs in combinations of five. So $C(n,5) \geq 21 = \dfrac{n!}{(n-5)!5!} \geq 21 \Rightarrow$

$\dfrac{n(n-1)(n-2)(n-3)(n-4)(n-5)!}{(n-5)!5!} \geq 21 \Rightarrow n^5 - 10n^4 + 35n^3 - 50n^2 + 24n - 2520 \geq 0$. By Descartes' Rule of signs, we know that there are 5, 3, or 1 positive zeros. (Ignore the negative zeros since the exercise requires a positive answer.) The prime factorization of 2520 is $2^3 \cdot 3^2 \cdot 5 \cdot 7$, from which we can determine the possible rational zeros. Using synthetic division, we find that 7 is a zero:

$$\begin{array}{r|rrrrrr} 7 & 1 & -10 & 35 & -50 & 24 & -2520 \\ & & 7 & -21 & 98 & 336 & 2520 \\ \hline & 1 & -3 & 14 & 48 & 360 & 0 \end{array}$$

So, $(n-7)(n^4 - 3n^3 + 14n + 48 - 360) \geq 0.$ Continuing the process, we find that there are no rational zeros for the second factor. So the radio station needs seven songs to start its programming with five songs every day for 21 days without using the same five songs on any two days.

71. There are $C(30,2) = \dfrac{30!}{(30-2)!2!)} = 435$ lines.

72. There are $C(21,3) = \dfrac{21!}{(21-3)!3!)} = 1330$ triangles.

73. $\dbinom{m+1}{m-1} = 3! \Rightarrow \dfrac{(m+1)!}{(m-1)!((m+1)-(m-1))!} = 6 \Rightarrow \dfrac{(m+1)!}{(m-1)!2!} = 6 \Rightarrow \dfrac{(m+1)m(m-1)!}{(m-1)!} = 12 \Rightarrow$

$m^2 + m - 12 = 0 \Rightarrow (m+4)(m-3) = 0 \Rightarrow m = 3 \text{ or } m = -4$

Reject the negative solution because factorials are not defined for negative numbers. The solution is $\{3\}$.

74. $6\dbinom{n-1}{2} = \dbinom{n+1}{4} \Rightarrow \dfrac{6(n-1)!}{2!(n-3)!} = \dfrac{(n+1)!}{4!(n-3)!} \Rightarrow 3(n-1)! = \dfrac{(n+1)!}{24} \Rightarrow 72 = \dfrac{(n+1)!}{(n-1)!} \Rightarrow \dfrac{(n+1)n(n-1)!}{(n-1)!} = 72 \Rightarrow$

$n^2 + n - 72 = 0 \Rightarrow (n+9)(n-8) = 0 \Rightarrow n = -9 \text{ or } n = 8$

Reject the negative solution because factorials are not defined for negative numbers. The solution is $\{8\}$.

75. $2\binom{n-1}{2}=\binom{n}{3}\Rightarrow\frac{2(n-1)!}{2!(n-3)!}=\frac{n!}{3!(n-3)!}\Rightarrow\frac{(n-1)(n-2)(n-3)!}{(n-3)!}=\frac{n(n-1)(n-2)(n-3)!}{6(n-3)!}\Rightarrow$
$6(n-1)(n-2)=n(n-1)(n-2)\Rightarrow n=6$

76. $\frac{4}{3}\binom{k}{2}=\binom{k+1}{3}\Rightarrow\frac{4(k!)}{3(2!)(k-2)!}=\frac{(k+1)!}{3!(k-2)!}\Rightarrow\frac{4k(k-1)(k-2)!}{6(k-2)!}=\frac{(k+1)k(k-1)(k-2)!}{6(k-2)!}\Rightarrow$
$4k(k-1)=(k+1)k(k-1)\Rightarrow k+1=4\Rightarrow k=3$

77. Using the Binomial Theorem, $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=64$ is the sum of the coefficients in the nth row of the expansion of $(x+y)^n$. If $x = 1$ and $y = 1$, then

$(1+1)^n=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=64\Rightarrow 2^n=64\Rightarrow n=6$.

78. Using the Binomial Theorem,

$\binom{k}{1}+\binom{k}{2}+\binom{k}{3}+\cdots+\binom{k}{k-1}=126\Rightarrow$

$\binom{k}{0}+\binom{k}{1}+\binom{k}{2}+\binom{k}{3}+\cdots+\binom{k}{k-1}+\binom{k}{k}=126+\binom{k}{0}+\binom{k}{k}=126+\frac{k!}{0!(k-0)!}+\frac{k!}{k!(k-k)!}=128$

is the sum of the coefficients in the kth row of the expansion of $(x+y)^n$. If $x = 1$ and $y = 1$, then

$(1+1)^k=\binom{k}{0}+\binom{k}{1}+\binom{k}{2}+\cdots+\binom{k}{k}=128\Rightarrow 2^k=128\Rightarrow k=7$.

79. $\binom{m}{4}=\binom{m}{5}\Rightarrow\frac{m!}{(m-4)!4!}=\frac{m!}{(m-5)!5!}\Rightarrow\frac{5!}{4!}=\frac{(m-4)!}{(m-5)!}\Rightarrow 5=\frac{(m-4)(m-5)!}{(m-5)!}\Rightarrow 5=m-4\Rightarrow m=9$

80. $\binom{n}{7}=\binom{n}{3}\Rightarrow\frac{n!}{(n-7)!7!}=\frac{n!}{(n-3)!3!}\Rightarrow\frac{(n-3)!}{(n-7)!}=\frac{7!}{3!}\Rightarrow\frac{(n-3)(n-4)(n-5)(n-6)(n-7)!}{(n-7)!}=\frac{7\cdot6\cdot5\cdot4\cdot3!}{3!}\Rightarrow$
$(n-3)(n-4)(n-5)(n-6)=840\Rightarrow n^4-18n^3+119n^2-342n-480=0$

The possible rational zeros are $\pm1, \pm2, \pm3, \pm4, \pm5, \pm6, \pm8, \pm10, \pm12, \pm15, \pm16, \pm20, \pm24, \pm30,$
$\pm32, \pm40, \pm48, \pm60, \pm80, \pm96, \pm120, \pm160, \pm240,$ and ±480.

Using synthetic division, we find that 10 is a zero:

$$\begin{array}{r|rrrrr} 10 & 1 & -18 & 119 & -342 & -480 \\ & & 10 & -80 & 390 & 480 \\ \hline & 1 & -8 & 39 & 48 & 0 \end{array}$$

Another zero is -1:

$$\begin{array}{r|rrrr} -1 & 1 & -8 & 39 & 48 \\ & & -1 & 9 & -48 \\ \hline & 1 & -9 & 48 & 0 \end{array}$$

The zeros of the depressed equation are complex. Reject -1 because factorials are not defined for negative numbers. The solution is $\{10\}$.

11.6 Critical Thinking/Discussion/Writing

81. There are $5\cdot5=25$ terms of the form x^ny^m if n and m are any integers such that $1\le n\le5$ and $1\le m\le5$.

82. There are two terms in the first factor and three terms in the second factor, so there are $2\cdot3=6$ terms.

11.6 Maintaining Skills

83. $E = \{2, 5, 7\} \Rightarrow n(E) = 3$

84. $E = \{-3, -1\} \Rightarrow n(E) = 2$

85. $E = \{0\} \Rightarrow n(E) = 1$

86. $E = \varnothing \Rightarrow n(E) = 0$

For exercises 87–90, $A = \{1, 2, 3, 5\}$ and $B = \{3, 4, 5, 6, 7\}$.

87. $n(A) = 4,\ n(B) = 5$

88. $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$
$n(A \cup B) = 7$

89. $A \cap B = \{3, 5\}$
$n(A \cap B) = 2$

90. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
$7 = 4 + 5 - 2$ True

For exercises 91–94, $A = \{a, b, c\}$ and $B = \{2, 4, 6, 8\}$.

91. $n(A) = 3,\ n(B) = 4$

92. $A \cup B = \{a, b, c, 2, 4, 6, 8\}$
$n(A \cup B) = 7$

93. $A \cap B = \varnothing$
$n(A \cap B) = 0$

94. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
$7 = 3 + 4 - 0$ True

11.7 Probability

11.7 Practice Problems

1. a. $E_1 = \{1, 3, 5\},\ P(E_1) = \frac{3}{6} = \frac{1}{2}$

b. $E_2 = \{5, 6\},\ P(E_2) = \frac{2}{6} = \frac{1}{3}$

2. $S = \{(H, H), (H, T), (T, H), (T, T)\}$
$E = \{(H, T), (T, H)\}$
$P(E) = \frac{2}{4} = \frac{1}{2}$

3. $E = \{(6, 1), (5, 2), (4, 3), (3, 4), (2, 5), (6, 1)\}$
There are 36 possible outcomes, so
$P(E) = \frac{6}{36} = \frac{1}{6}$

4. Since the order in which the numbers are selected is not important, the sample space S consists of all sets of 6 numbers that can be selected from 50 numbers. So,

$$n(S) = C(50, 6) = \frac{50!}{(50-6)!6!} = \frac{50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 \cdot 45}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 15{,}890{,}700$$

$$P(\text{winning the lottery}) = \frac{1}{15{,}890{,}700}$$

5. $P(\text{jack or king}) = P(\text{jack}) + P(\text{king}) = \frac{4}{52} + \frac{4}{52} = \frac{8}{52} = \frac{2}{13}$

6. $$P(\text{Brandy is not selected}) = \frac{C(9,3)}{C(10,3)} = \frac{9!}{3!6!} \cdot \frac{3!7!}{10!} = \frac{7}{10}$$

$P(\text{Brandy is selected})$
$$= 1 - P(\text{Brandy is not selected}) = 1 - \frac{7}{10} = \frac{3}{10}$$

7. The probability that a student selected at random is a male is $\frac{45{,}143}{82{,}074} \approx 0.55$. The probability that a student selected at random attends Sacramento College is $\frac{20{,}878}{82{,}074} \approx 0.25$. The probability that a student selected at random is a male attending Sacramento College is $\frac{9040}{82{,}074} \approx 0.11$. So, the probability that a student selected at random is a male or attends Sacramento College is approximately $0.55 + 0.25 - 0.11 = 0.69$.

11.7 Basic Concepts and Skills

1. If no outcome of an experiment results more often than any other outcome, then the outcomes are said to be equally likely.

2. A *certain* event has probability 1, and an *impossible* event has probability 0.

3. If it is impossible for two events to occur simultaneously, then the events are said to be mutually exclusive.

4. If the probability that an event occurs is 0.7, then the probability that the event does not occur is $\underline{1 - 0.7 = 0.3}$.

5. True (if "between 0 and 1" includes 0 and 1.)

6. True

7. S = {(Wendy's, McDonald's, Burger King), (Wendy's, Burger King, McDonald's), (McDonald's, Wendy's, Burger King), (McDonald's, Burger King, Wendy's), (Burger King, McDonald's, Wendy's), (Burger King, Wendy's, McDonald's)}

8. S = {(hamburger, Wendy's), (hamburger, McDonald's), (hamburger, Burger King), (cheeseburger, Wendy's), (cheeseburger, McDonald's), (cheeseburger, Burger King)}.

9. S = {{*Thriller*, *The Wall*}, {*Thriller*, *Eagles: Their Greatest Hits*}, {*Thriller*, *Led Zeppelin IV*}, {*The Wall*, *Eagles: Their Greatest Hits*}, {*The Wall*, *Led Zeppelin IV*}, {*Eagles: Their Greatest Hits*, *Led Zeppelin IV*}}

10. S = {{regular salted, regular unsalted, barbecue}, {regular salted, regular unsalted, cheddar cheese}, {regular salted, regular unsalted, sour cream and onion}, {regular salted, barbecue, cheddar cheese}, {regular salted, barbecue, sour cream and onion}, {regular salted, cheddar cheese, sour cream and onion}, {regular unsalted, barbecue, cheddar cheese}, {regular unsalted, barbecue, sour cream and onion}, {regular unsalted, cheddar cheese, sour cream and onion}, {barbecue, cheddar cheese, sour cream and onion}}

11. S = {(white, male), (white, female), (African-American, male), (African-American, female), (Native American, male), (Native American, female), (Asian, male), (Asian, female), (other, male), (other, female), (multiracial, male), (multiracial, female)}

12. S = {(white, 1), (white, 2), (white, 3), (white, 4), (white, 5), (white, 6), (red, 1), (red, 2), (red, 3), (red, 4), (red, 5), (red, 6), (green, 1), (green, 2), (green, 3), (green, 4), (green, 5), (green, 6)}

13. $P = 0.$ **14.** $P = 1$

15. $P = 1$ **16.** $P = 0$

17. experimental **18.** theoretical

19. theoretical **20.** experimental

21. experimental **22.** experimental

23. theoretical **24.** experimental

25. experimental **26.** experimental

27. An event that is very likely to happen has probability 0.999. An event that will surely happen has probability 1. An event that is a rare event has probability 0.001. An event that is equally likely to happen or not happen has probability 0.5. An event that will never happen has probability 0.

28. {the sure event},{at least one head},{at least two heads}, {exactly two heads}

29. $E = \{1, 6\}, P(E) = \dfrac{1}{3}$

30. $E = \{5\}, P(E) = \dfrac{1}{6}$

31. $E = \{5, 6\}, P(E) = \dfrac{1}{3}$

32. $E = \{3, 6\}, P(E) = \dfrac{1}{3}$

33. $E = \{2, 4, 6\}, P(E) = \dfrac{1}{2}$

34. $E = \varnothing, P(E) = 0$

35. E = {club queen, spade queen, heart queen, diamond queen}, $P(E) = \dfrac{1}{13}$

36. E = {club ace, spade ace, heart ace, diamond ace}, $P(E) = \dfrac{1}{13}$

37. E = {heart ace, heart 2, heart 3, heart 4, heart 5, heart 6, heart 7, heart 8, heart 9, heart 10, heart jack, heart queen, heart king},
$P(E) = \dfrac{1}{4}$

38. E = {heart ace, heart king}, $P(E) = \dfrac{1}{26}$

39. E = {heart jack, heart queen, heart king, diamond jack, diamond queen, diamond king, club jack, club queen, club king, spade jack, spade queen, spade king}, $P(E) = \dfrac{3}{13}$

40. E = {heart ace, heart 2, heart 3, heart 4, heart 5, heart 6, heart 7, heart 8, heart 9, heart 10},
$P(E) = \dfrac{5}{26}$

41. $E = \{8+3\}, P(E) = \dfrac{1}{10}$

42. $E = \varnothing, P(E) = 0$

43. $E = \{0+3\}, P(E) = \dfrac{1}{10}$

44. $E = \{6+3, 7+3, 8+3, 9+3\}, P(E) = \dfrac{2}{5}$

45. $E = \{1+3, 3+3, 5+3, 7+3, 9+3\}, P(E) = \dfrac{1}{2}$

46. $E = \{0+3, 2+3, 4+3, 6+3, 8+3\}, P(E) = \dfrac{1}{2}$

11.7 Applying the Concepts

47. The probability the Tony will not like his blind date is 1 − 0.3 = 0.7

48. The probability of drawing a card that is not a spade is 1 − 0.25 = 0.75

49. $P = \dfrac{20}{100} = \dfrac{1}{5}$

50. $P = \dfrac{85}{100} = \dfrac{17}{20}$

51. 6% of the Marines were women, so 94% were men. The probability that a Marine chosen at random is a man is $\dfrac{94}{100} = \dfrac{47}{50}$.

52. The probability that a person over 65 will go to a movie is $1/5$, so the probability that a person over 65 will not go to a movie is $4/5$.

53. If you have 10 tickets and there is only one prize, then the probability of winning is $\dfrac{10}{125} = \dfrac{2}{25}$. If there are two prizes, then there are $C(125, 2) = 7750$ possible winning combinations, and you have $C(10, 2) = 45$ possible winning tickets. So the probability of winning is $\dfrac{45}{7750} = \dfrac{9}{1550}$.

54. There are $C(8,2) = 28$ possible combinations. There are $C(4, 2) = 6$ possible combinations of two men, and the same number of combinations of two women. So there are 28 − 12 = 16 possible combinations of one man and one woman. Therefore, the probability of selecting a man and a woman for the committee is $\dfrac{16}{28} = \dfrac{4}{7}$.

55. a. There are $2^2 = 4$ possible combinations (bb, bg, gb, gg). The probability of having two boys is $1/4$.

b. The probability of having two girls is $1/4$.

c. The probability of having one boy and one girl is $1/2$.

56. a. There are $2^3 = 8$ possible combinations (bbb, bbg, bgb, bgg, gbb, gbg, ggb, ggg). The probability of having all girls is $1/8$.

b. There are four combinations of at least two girls, so the probability of having at least two girls is $\dfrac{4}{8} = \dfrac{1}{2}$.

c. There are seven combinations of at most two girls, so the probability of having at most two girls is $7/8$.

d. There are three combinations of exactly two girls, so the probability of having exactly two girls is $3/8$.

57. a. Out of the 4440 families, 2370 families own two cars, so the probability of a family owning two cars is $\dfrac{2370}{4440} = \dfrac{79}{148}$.

b. Out of the 4440 families, 37 own no cars, 1256 own one car, and 2370 own two cars. So the probability of a family owning at most two cars is $\dfrac{3663}{4440} = \dfrac{33}{40}$.

c. Out of the 4440 families, none own six cars, so the probability of a family owning six cars is 0.

d. All of the families own fewer than six cars, so the probability of a family owning at most six cars is 1.

e. Of the 4440 families, all but 37 own at least one car, so the probability of a family owning at least one car is $\dfrac{4403}{4340} = \dfrac{119}{120}$.

58. The probability of a mammogram not being normal is $1 - \dfrac{199}{200} = \dfrac{1}{200}$.

59. The probability that a non-Hispanic white American has no lactose intolerance is $1-0.2=0.8=\frac{4}{5}$. The probability that an African-American, Asian, or Native American has no lactose intolerance is $1-0.75=0.25=1/4$.

60. a. There are $2^{10}=1024$ possible combinations of answers on the test. There is only one way to answer all the questions correctly, so the probability of scoring 100% is $\frac{1}{1024}$.

b. There are $C(10,9)=10$ possible ways to answer nine questions correctly, so the probability of scoring 90% is $\frac{10}{1024}=\frac{5}{512}$.

c. From parts (a) and (b), there are 11 possible ways to score at least 90%, so the probability of scoring at least 90% is $\frac{11}{1024}$.

61. a. There are $C(10,2)=10$ different combinations of people for the committee. There is only one combination of the two oldest members, so the probability of the committee consisting of the two oldest members is $1/10$.

b. There is only one combination of the oldest and youngest members, so the probability is $1/10$.

62. Of the 73.7 million children, there are 24.5 + 25.1 = 49.6 million who are at least 6 years old. The probability that a randomly chosen child is at least 6 years old is $\frac{49.6}{73.7}=0.6730.$

63.

Number of people who are	Depressed according to the test	Normal according to the test
Actually depressed	90	10
Actually normal	135	765

If 10% of the population suffers from depression, then we expect that 10% of the 1000 people tested = 100 people will suffer from depression and 900 will not.

Of the 100 people, the test will work for 90% of them, or 90 people, and the remaining 10 people are normal according to the test. Of the 900 people who don't suffer from depression, the test works for 85% of them, or 765 people. The remaining 135 are depressed according to the test.

64. Sample points: $\{M_1M_2N, M_1NM_2, M_2M_1N, M_2NM_1, NM_1M_2, NM_2M_1\}$. There are four ways for Deshawn to win out of the six possibilities, so the probability that he will win is $\frac{4}{6}=\frac{2}{3}$.

65. The possible combinations are {bbbb, bbbg, bbgg, bggg, gggg}. (Birth order is not important.) So, there are two ways to have three children of one gender and one of the other gender, while there is only one way to have two children of each gender. Therefore, it is more likely to have three children of one gender and one of the other.

66. Eleven out of 100 patients are admitted for surgery and fifteen out of 100 are admitted for obstetrics, while 3 out of 100 are admitted for both. So the probability that a patient is admitted for either obstetrics or surgery is $0.11+0.15-0.03=0.23=23\%$.

67. a. Out of 180 people, 50 people received the placebo, a total of 56 people had no pain relief, while 34 people received the placebo and had no pain relief. So the probability that a person either received the placebo or had no pain relief is $\frac{50}{180}+\frac{56}{180}-\frac{34}{180}=\frac{2}{5}$.

b. Out of 180 people, 70 people received the new medicine, a total of 69 people had complete pain relief, while 40 people who received the new medicine also had complete pain relief. So, the probability that a person either received the new medicine or had complete pain relief is $\frac{70}{180}+\frac{69}{180}-\frac{40}{180}=\frac{99}{180}$.

11.7 Beyond the Basics

68. a. In a group of two people, the probability that they both have the same birthday is $1/365$, since there are 365 days in the year and they have one day in common.

b. For a group of three people, first find the probability that no two will have the same birthday: The first person's birthday can occur on any of the 365 days in the year. Then the second person's birthday can occur on 364 days, and the third person's birthday can occur on 363 days. So the probability that no two people in a group of three will have the same birthday is

$$\frac{P(365,3)}{365^3} = \frac{365 \bullet 364 \bullet 363}{365^3} = \frac{132{,}132}{133{,}225}.$$

Then, the probability that any two people in a group of three will have the same birthday is $1 - \frac{132{,}132}{133{,}225} = \frac{1093}{133{,}225}$.

c. For a group of 50 people, the probability of no birthday match is

$$\frac{P(365,50)}{365^{50}} = \frac{P(364,49)}{365^{49}} = \frac{\frac{364!}{(364-49)!}}{365^{49}}.$$
$$= \frac{\frac{364!}{315!}}{365^{49}}$$

Then the probability of a birthday match is

$$1 - \frac{\frac{364!}{315!}}{365^{49}} = \frac{365^{49} - \frac{364!}{315!}}{365^{49}} \approx 0.97$$

69. To find the probability that a number between 1 and 1,000,000 does not contain the digit 3, note that each place value through the hundred-thousand's place can contain one of nine digits. So the probability that a number between 1 and 1,000,000 will not contain the digit 3 is $\frac{9^6}{1{,}000{,}000} = \frac{531{,}441}{1{,}000{,}000}$. So the probability that a number between 1 and 1,000,000 will contain the digit 3 is

$$1 - \frac{531{,}441}{1{,}000{,}000} = \frac{468{,}559}{1{,}000{,}000}.$$

70. The first letter can be placed in any of the four envelopes, the second letter can be placed in any of the remaining three envelopes, the third letter can be placed in either of the remaining two envelopes, and the fourth letter will be placed in the last envelope. So there are $4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways to place the letters in the envelopes. The probability of placing letters in the correct envelopes is $1/24$.

71. Because one of the digits is repeated, there are $\frac{4!}{2!1!1!} = 12$ different ways to arrange the digits. The probability that Maggie will dial the correct number is $1/12$.

11.7 Critical Thinking/Discussion/Writing

72. The probability that a person chosen at random in the survey would report his or her highest level of education as high school graduation is $\frac{59{,}840{,}000}{187{,}000{,}000} = \frac{8}{25}$.

73. If there are x dark caramels in the box, then there are $2x$ light caramels, and a total of $3x$ caramels in the box. So the probability of choosing a dark caramel is $\frac{x}{3x} = \frac{1}{3}$.

74. If a single die is rolled two times, then there are $6 \cdot 6 = 36$ possible outcomes. If the first number rolled is n, then there are $6 - n$ possible rolls in which the second roll will be greater than the first roll. So there are $\sum_{n=1}^{5} 6 - n = 15$ possible rolls. The probability that the second roll will result in a number larger than the first roll is $\frac{15}{36} = \frac{5}{12}$.

Chapter 11 Review Exercises

1. $a_1 = 2(1) - 3 = -1, a_2 = 2(2) - 3 = 1,$
$a_3 = 2(3) - 3 = 3, a_4 = 2(4) - 3 = 5,$
$a_5 = 2(5) - 3 = 7$

2. $a_1 = \frac{1(1-2)}{2} = -\frac{1}{2}, a_2 = \frac{2(2-2)}{2} = 0,$
$a_3 = \frac{3(3-2)}{2} = \frac{3}{2}, a_4 = \frac{4(4-2)}{2} = 4,$
$a_5 = \frac{5(5-2)}{2} = \frac{15}{2}$

3. $a_1 = \frac{1}{2(1)+1} = \frac{1}{3}, a_2 = \frac{2}{2(2)+1} = \frac{2}{5},$
$a_3 = \frac{3}{2(3)+1} = \frac{3}{7}, a_4 = \frac{4}{2(4)+1} = \frac{4}{9},$
$a_5 = \frac{5}{2(5)+1} = \frac{5}{11}$

4. $a_1 = (-2)^{1-1} = 1, a_2 = (-2)^{2-1} = -2,$
$a_3 = (-2)^{3-1} = 4, a_4 = (-2)^{4-1} = -8,$
$a_5 = (-2)^{5-1} = 16$

5. This is an arithmetic sequence with a common difference of -2. $a_n = 32 - 2n$ for $n \ge 1$.

6. This is a geometric sequence with a common ratio of -2. $a_n = -(-2)^{n-1}$ for $n \ge 1$.

7. $\frac{9!}{8!} = 9$

8. $\frac{10!}{7!} = \frac{10 \cdot 9 \cdot 8 \cdot 7!}{7!} = 720$

9. $\frac{(n+1)!}{n!} = n+1$

10. $\frac{n!}{(n-2)!} = \frac{n(n-1)(n-2)!}{(n-2)!} = n^2 - n$

11. $\sum_{k=1}^{4} k^3 = 1^3 + 2^3 + 3^3 + 4^3 = 100$

12. $\sum_{j=1}^{5} \frac{1}{2j} = \frac{1}{2(1)} + \frac{1}{2(2)} + \frac{1}{2(3)} + \frac{1}{2(4)} + \frac{1}{2(5)} = \frac{137}{120}$

13. $\sum_{k=1}^{7} \frac{k+1}{k}$
$= \frac{1+1}{1} + \frac{2+1}{2} + \frac{3+1}{3} + \frac{4+1}{4} + \frac{5+1}{5}$
$+ \frac{6+1}{6} + \frac{7+1}{7} = \frac{1343}{140}$

14. $\sum_{k=1}^{5} (-1)^k 3^{k+1}$
$= (-1)^1 3^{1+1} + (-1)^2 3^{2+1} + (-1)^3 3^{3+1}$
$+ (-1)^4 3^{4+1} + (-1)^5 3^{5+1} = -549$

15. $\sum_{k=1}^{50} \frac{1}{k}$ **16.** $-\sum_{k=1}^{5} (-2)^k$

17. The sequence is arithmetic. $a_1 = 11, d = -5$

18. The sequence is arithmetic. $a_1 = \frac{2}{3}, d = \frac{1}{6}$

19. The sequence is not arithmetic.

20. The sequence is arithmetic. $a_1 = -2, d = 3$

21. $a_n = 3n$ **22.** $a_n = 4n + 1$

23. $a_n = x + n - 1$ **24.** $a_n = (2n+1)x$

25. $a_3 = a_1 + d(3-1) \Rightarrow 7 = a_1 + 2d$
$a_8 = a_1 + d(8-1) \Rightarrow 17 = a_1 + 7d$
$\begin{cases} a_1 + 2d = 7 \\ a_1 + 7d = 17 \end{cases} \Rightarrow -5d = -10 \Rightarrow d = 2, a_1 = 3$
$a_n = 3 + 2(n-1) = 2n + 1$

26. $a_5 = a_1 + d(5-1) \Rightarrow -16 = a_1 + 4d$
$a_{20} = a_1 + d(20-1) \Rightarrow -46 = a_1 + 19d$
$\begin{cases} a_1 + 4d = -16 \\ a_1 + 19d = -46 \end{cases} \Rightarrow -15d = 30 \Rightarrow$
$d = -2, a_1 = -8$
$a_n = -8 - 2(n-1) = -2n - 6$

27. $d = 2, a_1 = 7, a_n = 37 \Rightarrow$
$37 = 7 + 2(n-1) \Rightarrow n = 16$
$S = 16\left(\frac{7+37}{2}\right) = 352$

28. $d = \frac{1}{4}, a_1 = \frac{1}{4}, a_n = 15 \Rightarrow$
$15 = \frac{1}{4} + \frac{1}{4}(n-1) \Rightarrow n = 60$
$S = 60\left(\frac{1/4 + 15}{2}\right) = \frac{915}{2}$

29. $d = 5, a_1 = 3, n = 40 \Rightarrow$
$a_n = 3 + 5(40-1) \Rightarrow a_n = 198$
$S = 40\left(\frac{3+198}{2}\right) = 4020$

30. $d = 0.5, a_1 = -6, n = 60 \Rightarrow$
$a_n = -6 + 0.5(60-1) \Rightarrow a_n = 23.5$
$S = 60\left(\frac{-6+23.5}{2}\right) = 525$

31. The sequence is geometric. $a_1 = 4, r = -2$

32. The sequence is geometric. $a_1 = \frac{1}{5}, r = \frac{1}{2}$

33. The sequence is not geometric.

34. The sequence is geometric. $a_1 = \frac{1}{2}, r = \frac{1}{2}$

35. $a_1 = 16, r = -\frac{1}{4}, a_n = 16 \cdot \left(-\frac{1}{4}\right)^{n-1} = \frac{(-1)^{n-1}}{4^{n-3}}$

36. $a_1 = -\frac{5}{6}, r = \frac{2}{5},$

$a_n = \left(-\frac{5}{6}\right)\left(\frac{2}{5}\right)^{n-1} = -\frac{1}{3}\left(\frac{2}{5}\right)^{n-2}$

37. $a_{10} = a_1 r^{10-1} = 2 \cdot 3^9 = 39{,}366$

38. $a_{12} = a_1 r^{12-1} = -2\left(\frac{3}{2}\right)^{11} = -\frac{177{,}147}{1024}$

39. $r = \frac{1/5}{1/10} = 2; S_{12} = \frac{(1/10)\left(1-2^{12}\right)}{1-2} = \frac{819}{2}$

40. $r = \frac{-1}{2} = -\frac{1}{2}; S_{12} = \frac{2\left(1-(-1/2)^{10}\right)}{1-(-1/2)} = \frac{341}{256}$

41. $r = \frac{1/6}{1/2} = \frac{1}{3}; S = \frac{1/2}{1-(1/3)} = \frac{3}{4}$

42. $r = \frac{-2}{-5} = \frac{2}{5}; S = \frac{-5}{1-(2/5)} = -\frac{25}{3}$

43. $a_1 = \frac{3}{5}, r = \frac{3}{5}; \sum_{i=1}^{\infty}\left(\frac{3}{5}\right)^i = \frac{3/5}{1-3/5} = \frac{3}{2}$

44. $a_1 = -\frac{7}{4}, r = -\frac{1}{4};$

$\sum_{i=1}^{\infty} 7\left(-\frac{1}{4}\right)^i = \frac{-7/4}{1-(-1/4)} = -\frac{7}{5}$

45. For $n = 1, \sum_{k=1}^{1} 2^k = 2 = 2^{1+1} - 2$ is true.

Assume that it is true for

$n = m: \sum_{k=1}^{m} 2^k = 2^{m+1} - 2$.

Then for $n = m+1$,

$$\sum_{k=1}^{m+1} 2^k = \left(\sum_{k=1}^{m} 2^k\right) + 2^{m+1} = 2^{m+1} - 2 + 2^{m+1}$$
$$= 2(2^{m+1}) - 2 = 2^{m+2} - 2$$

which is exactly the statement for $n = m+1$. Therefore the formula is true for all natural numbers.

46. For $n = 1, \sum_{k=1}^{1} k(k+1) = 2 = \frac{1(1+1)(1+2)}{3}$ is true.

Assume that it is true for

$n = m: \sum_{k=1}^{m} k(k+1) = \frac{m(m+1)(m+2)}{3}$.

Then for $n = m+1$,

$$\sum_{k=1}^{m+1} k(k+1) = \left(\sum_{k=1}^{m} k(k+1)\right) + (m+1)(m+2)$$
$$= \frac{m(m+1)(m+2)}{3} + (m+1)(m+2)$$
$$= \frac{m(m+1)(m+2) + 3(m+1)(m+2)}{3}$$
$$= \frac{(m+1)(m+2)(m+3)}{3}$$

which is exactly the statement for $n = m+1$. Therefore the formula is true for all natural numbers.

47. $\binom{12}{7} = \frac{12!}{7!(12-7)!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7!}{7!5!}$

$= \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 792$

48. $\binom{11}{0} = \frac{11!}{0!(11-0)!} = \frac{11!}{11!} = 1$

49. $(x-3)^4$

$= x^4 + 4(-3)x^3 + 6(-3)^2 x^2 + 4(-3)^3 x + (-3)^4$

$= x^4 - 12x^3 + 54x^2 - 108x + 81$

50. $\left(\frac{x}{2} + 2\right)^6$

$$= \left(\frac{x}{2}\right)^6 + 6(2)\left(\frac{x}{2}\right)^5 + 15\left(2^2\right)\left(\frac{x}{2}\right)^4$$
$$+ 20\left(2^3\right)\left(\frac{x}{2}\right)^3 + 15\left(2^4\right)\left(\frac{x}{2}\right)^2$$
$$+ 6\left(2^5\right)\left(\frac{x}{2}\right)^6 + 2^6$$
$$= \frac{x^6}{64} + \frac{3x^5}{8} + \frac{15x^4}{4} + 20x^3 + 60x^2 + 96x + 64$$

51. The term containing x^5 is the eighth term.

$\binom{12}{5} x^5 (2^7) = \frac{12!}{5!(12-5)!} x^5 (2^7) = 101{,}376 x^5$

52. The term containing x^7 is the sixth term.

$$\binom{12}{7}x^7(2y)^5 = \frac{12!}{7!(12-7)!}x^7(2y)^5$$
$$= 25,344x^7y^5$$

53. If no repetitions are allowed, using the Fundamental Counting Principle, there are $4\cdot 3\cdot 2\cdot 1 = 24$ different numbers that can be written.

54. Since order is important, find the number of permutations of 10 taken 3 at a time:

$$P(10,3) = \frac{10!}{(10-3)!} = 720.$$

So, ten horses can finish in first, second, and third place in 720 different ways.

55. Since order is important, find the number of permutations of 7 taken 7 at a time:

$$P(7,7) = \frac{7!}{(7-7)!} = 5040.$$

There are 5040 different ways that seven people can line up.

56. There are 7 ways to enter the building and 6 ways to leave, so there are $7\cdot 6 = 42$ ways to enter and leave by a different entrance.

57. The five movies can be listed in $5! = 120$ different ways.

58. There are $C(4, 1)$ amounts consisting of one coin, $C(4, 2)$ amounts consisting of two coins, $C(4, 3)$ amounts consisting of three coins, and $C(4, 4)$ amounts consisting of four coins. So there are

$$C(4,1)+C(4,2)+C(4,3)+C(4,4)$$
$$= \frac{4!}{(4-1)!1!}+\frac{4!}{(4-2)!2!}+\frac{4!}{(4-3)!3!}+\frac{4!}{(4-4)!4!}$$

$= 15$ different amounts.

59. Order is not important, so find the number of combinations of three candies from a group of ten. There $C(10,3) = \frac{10!}{3!(10-3)!} = 120$ ways to choose three candies from a box of ten candies.

60. There are 12 face cards in a standard 52-card deck, so there are $C(12,2) = \frac{12!}{2!(12-2)!} = 66$ different ways that two face cards can be drawn.

61. There are $C(12, 2) = 66$ ways to choose two shirts from 12, and $C(8, 3) = 56$ ways to choose three pairs of pants from eight. So, there are $66\cdot 56 = 3696$ ways to choose two shirts and three pairs of pants.

62. $C(8, 2) = 28$ doubles teams can be formed from eight tennis players.

63. There are nine letters, so $n = 9$. There are two R's, three E's, and two T's. So there are $\frac{9!}{2!3!2!1!1!} = 15,120$ distinguishable ways to arrange the letters.

64. There are $6\cdot 5\cdot 4\cdot 3 = 360$ different arrangements.

65. Pair the two coins that appear together as one so that there are three positions to fill. Then, (with the paired coins counted as one) there are $3! = 6$ ways to fill these positions. Since the paired coins can be inserted into any position in $2! = 2$ ways, there are $3!2! = 6\times 2 = 12$ allowable ways to arrange the coins. The total number of ways to arrange the coins is $4! = 24$, so the probability of arranging the coins so that the two most recent dates will be next to each other is $\frac{12}{24} = \frac{1}{2}$.

66. There are ten stones in all, including three black stones, so the probability of selecting a black stone is $3/10$.

67. There are six letters, so there are $C(6, 2) = 15$ combinations of two letters. There are four consonants, so there are $C(4, 2) = 6$ combinations of two consonants. So the probability of choosing two consonants is $6/15 = 2/5$.

68. There are 30 people and two ways to win, so the probability of winning is $\frac{2}{30} = \frac{1}{15}$.

69. There are $C(8, 2) = 28$ ways to choose two people from the group of either. There are five ways to choose one man and three ways to choose one woman, so there are $5\cdot 3 = 15$ ways to choose one man and one woman. So the probability of choosing one man and one woman is $15/28$.

70. If 32% of the email received is junk mail, then 1 − 0.32 = 68% of the email received is not junk mail. The probability of receiving email that is not junk mail is $\frac{68}{100} = \frac{17}{25}$.

71. There are 13 clubs, so there are $C(13, 2) = 78$ ways to choose two clubs. There are $C(52, 2) = 1326$ ways to choose two cards from the entire deck, so the probability of choosing two clubs is $\frac{78}{1326} = \frac{1}{17}$.

72. If 30 out of 500 batteries are defective, then 470 are not defective, and the probability that a randomly selected battery is not defective is $\frac{470}{500} = \frac{47}{50}$.

73. a. $1/9$ **b.** $4/9$ **c.** $5/9$

74. a. There are 13 items in the dryer, including eight socks. The probability of pulling a sock is $8/13$.

b. The probability of pulling a nightgown is $3/13$

Chapter 11 Practice Test A

1. $a_1 = 3(5-4(1)) = 3, a_2 = 3(5-4(2)) = -9,$
$a_3 = 3(5-4(3)) = -21, a_4 = 3(5-4(4)) = -33,$
$a_5 = 3(5-4(5)) = -45$.
The sequence is arithmetic.

2. $a_1 = -3(2^1) = -6, a_2 = -3(2^2) = -12,$
$a_3 = -3(2^3) = -24, a_4 = -3(2^4) = -48,$
$a_5 = -3(2^5) = -96$. The sequence is geometric.

3. $a_1 = -2, a_2 = 3a_1 + 5 = 3(-2) + 5 = -1,$
$a_3 = 3a_2 + 5 = 3(-1) + 5 = 2,$
$a_4 = 3a_3 + 5 = 3(2) + 5 = 11,$
$a_5 = 3a_4 + 5 = 3(11) + 5 = 38$

4. $\frac{(n-1)!}{n!} = \frac{(n-1)!}{n(n-1)!} = \frac{1}{n}$

5. $a_7 = -3 + 6(4) = 21$

6. $a_8 = 13\left(-\frac{1}{2}\right)^7 = -\frac{13}{128}$

7. $a_1 = 1, a_{20} = 58$
$$\sum_{k=1}^{20}(3k-2) = 20\left(\frac{1+58}{2}\right) = 590$$

8. $a_1 = \frac{3}{8}, a_5 = \frac{3}{128}$
$$\sum_{k=1}^{5}\left(\frac{3}{4}\right)\left(2^{-k}\right) = \frac{\frac{3}{8}\left(1-\left(\frac{1}{2}\right)^5\right)}{1-\frac{1}{2}} = \frac{93}{128}$$

9. $a_1 = \frac{9}{50}; \sum_{k=1}^{\infty} 18\left(\frac{1}{100}\right)^k = \frac{9/50}{1-1/100} = \frac{2}{11}$

10. $\binom{13}{0} = \frac{13!}{0!(13-0)!} = 1$

11. $(1-2x)^4$
$= 1^4 + 4(1^3)(-2x) + 6(1^2)(-2x)^2 + 4(1)(-2x)^3 + (-2x)^4$
$= 16x^4 - 32x^3 + 24x^2 - 8x + 1$

12. The term containing x^1 is the fourth term.
$\binom{4}{1}(2x)^1(1)^3 = \frac{4!}{1!(4-1)!}(2x)^1(1)^3 = 8x$

13. There are $2^{10} = 1024$ different ways to answer every question on a ten-question true-false test.

14. $P(9,2) = \frac{9!}{(9-2)!} = 72$

15. $C(7,5) = \frac{7!}{(7-5)!5!} = 21$

16. There are $6 \cdot 10 = 60$ ways to fill the positions.

17. There are $26 \cdot 26 \cdot 10^4 = 6,760,000$ possible license plate numbers.

18. $\frac{5}{12}$ **19.** $\frac{5}{6}$

20. There are 18 coins and $C(18, 2) = 153$ ways to choose two coins. There are $C(7, 2) = 21$ ways to choose two quarters. So the probability of choosing two quarters is $\frac{21}{153} = \frac{7}{51}$.

Chapter 11 Practice Test B

1. $a_1 = 4(2(1)-3) = -4, a_2 = 4(2(2)-3) = 4,$
$a_3 = 4(2(3)-3) = 12, a_4 = 4(2(4)-3) = 20,$
$a_5 = 4(2(5)-3) = 28$. The sequence is arithmetic. The answer is C.

2. $a_1 = 2(4^1) = 8, a_2 = 2(4^2) = 32,$
$a_3 = 2(4^3) = 128, a_4 = 2(4^4) = 512,$
$a_5 = 2(4^5) = 2048$. The sequence is geometric. The answer is D.

3. $a_1 = 5, a_2 = 2a_1 + 4 = 2(5) + 4 = 14,$
$a_3 = 2a_2 + 4 = 2(14) + 4 = 32,$
$a_4 = 2a_3 + 4 = 2(32) + 4 = 68,$
$a_5 = 2a_4 + 4 = 2(68) + 4 = 140$.
The answer is A.

4. $\dfrac{(n+2)!}{(n+2)} = \dfrac{(n+2)(n+1)!}{(n+2)} = (n+1)!$.
The answer is C.

5. $a_8 = a_1 + 7d = -6 + 7(3) = 15$
The answer is A.

6. $a_{10} = 247\left(\dfrac{1}{3}\right)^9 = \dfrac{247}{19,683}$
The answer is B.

7. $a_1 = -3, a_{45} = 4(45) - 7 = 173$
$\displaystyle\sum_{k=1}^{45}(4k-7) = 45\left(\frac{-3+173}{2}\right) = 3825$
The answer is D.

8. $a_1 = \dfrac{8}{3}, r = 2; \displaystyle\sum_{k=1}^{5}\left(\frac{4}{3}\right)(2^k) = \frac{\frac{8}{3}\left(1-2^5\right)}{1-2} = \frac{248}{3}$.
The answer is A.

9. $a_1 = 8; \displaystyle\sum_{k=1}^{\infty}8(-0.3)^{k-1} = \frac{8}{1-(-0.3)} \approx 6.15$.
The answer is B.

10. $\dbinom{12}{11} = \dfrac{12!}{11!(12-11)!} = 12$
The answer is B.

11. $(3x-1)^4$
$= (3x)^4 + 4(3x)^3(-1) + 6(3x)^2(-1)^2 + 4(3x)(-1)^3 + (-1)^4$
$= 81x^4 - 108x^3 + 54x^2 - 12x + 1$
The answer is A.

12. The term containing x is the third term.
$\dbinom{3}{1}(2x)^1(3)^2 = \dfrac{3!}{1!(3-1)!}(2x)^1(9) = 54x$
The answer is D.

13. There are $4 \cdot 10 \cdot 3 = 120$ ways to choose a necklace, a pair of earrings, and a bracelet. The answer is C.

14. $P(7,3) = \dfrac{7!}{(7-3)!} = 210$
The answer is A.

15. $C(10,7) = \dfrac{10!}{7!(10-7)!} = 120$
The answer is D.

16. The first digit can be chosen in nine ways (note that the first digit cannot be zero), and the second digit can be chosen in nine ways. So there are $9 \cdot 9 = 81$ two-digit numbers that can be formed without using a digit more than once. The answer is B.

17. Since one CD is a "must buy", four CDs must be chosen from the remaining seven. There are $C(7,4) = \dfrac{7!}{4!(7-4)!} = 35$ ways to choose the CDs. The answer is D.

18. The answer is C.

19. There are $6 \cdot 6 = 36$ possible outcomes. The outcomes in which the sum is greater than 9 are {(4, 6), (6, 4), (5, 5), (5, 6), (6, 5) (6, 6)}, so the probability of getting a sum greater than 9 when two die are rolled is $\dfrac{6}{36} = \dfrac{1}{6}$.
The answer is A.

20. There are 52 − 13 = 39 non-spades in the deck, so the probability of choosing a card that is not a spade is $\dfrac{39}{52} = \dfrac{3}{4}$.
The answer is D.

Cumulative Review Exercises (Chapters P–11)

1. $|3x-8| = |x| \Rightarrow 3x-8 = x$ or $3x-8 = -x \Rightarrow$
$x = 4$ or $x = 2$. The solution is $\{2, 4\}$.

2. $x^2(x^2-5) = -4 \Rightarrow x^4 - 5x^2 + 4 = 0 \Rightarrow$
$(x^2-4)(x^2-1) = 0 \Rightarrow$
$(x-2)(x+2)(x-1)(x+1) = 0 \Rightarrow$
$x = 2$ or $x = -2$ or $x = 1$ or $x = -1$
The solution is $\{-2, -1, 1, 2\}$.

3. $\log_2(3x-5) + \log_2 x = 1 \Rightarrow$
$\log_2 x(3x-5) = 1 \Rightarrow 3x^2 - 5x = 2 \Rightarrow$
$3x^2 - 5x - 2 = 0 \Rightarrow (3x+1)(x-2) = 0 \Rightarrow$
$x = -\frac{1}{3}$ or $x = 2$
Reject the negative answer. The solution is $\{2\}$.

4. Let $u = e^x$. Then $e^{2x} - e^x - 2 = 0 \Rightarrow$
$u^2 - u - 2 = 0 \Rightarrow (u-2)(u+1) = 0 \Rightarrow$
$u = 2$ or $u = -1$. If $u = -1$, then
$-1 = e^x \Rightarrow \ln(-1) = x$, which is impossible,
so reject -1. If $u = 2$, then $2 = e^x \Rightarrow x = \ln 2$.
The solution is $\{\ln 2\}$.

5. $\frac{6}{x+2} = \frac{4}{x} \Rightarrow 6x = 4x + 8 \Rightarrow x = 4$

6. $2(x-8)^{-1} = (x-2)^{-1} \Rightarrow \frac{2}{x-8} = \frac{1}{x-2} \Rightarrow$
$2x - 4 = x - 8 \Rightarrow x = -4$

7. $\sqrt{x-3} = \sqrt{x} - 1 \Rightarrow \left(\sqrt{x-3}\right)^2 = \left(\sqrt{x}-1\right)^2 \Rightarrow$
$x - 3 = x - 2\sqrt{x} + 1 \Rightarrow 2 = \sqrt{x} \Rightarrow x = 4$

8. $x^2 + 4x \ge 0 \Rightarrow x(x+4) \ge 0$
Solving the associated equation, we have $x = 0$ or $x = -4$. The intervals to be tested are $(-\infty, -4], [-4, 0], [0, \infty)$.

Interval	Test point	Value of $x(x+4)$	Result
$(-\infty, -4]$	-5	5	+
$[-4, 0]$	-1	-3	−
$[0, \infty)$	1	5	+

The solution is $(-\infty, -4] \cup [0, \infty)$.

9. $18 \le x^2 + 6x \Rightarrow x^2 + 6x - 18 \ge 0$
Solving the associated equation, we have
$$x = \frac{-6 \pm \sqrt{36 - 4(-18)}}{2} = \frac{-6 \pm \sqrt{108}}{2} = \frac{-6 \pm 6\sqrt{3}}{2} = -3 \pm 3\sqrt{3}$$
The intervals to be tested are $\left(-\infty, -3-3\sqrt{3}\right]$, $\left[-3-3\sqrt{3}, -3+3\sqrt{3}\right], \left[-3+3\sqrt{3}, \infty\right)$.

Interval	Test point	Value of $x^2+6x-18$	Result
$\left(-\infty, -3-3\sqrt{3}\right]$	-10	22	+
$\left[-3-3\sqrt{3}, -3+3\sqrt{3}\right]$	0	-18	−
$\left[-3+3\sqrt{3}, \infty\right)$	3	9	+

The solution is
$\left(-\infty, -3-3\sqrt{3}\right] \cup \left[-3+3\sqrt{3}, \infty\right)$.

10. $\begin{cases} 5x + 4y = 6 \\ 4x - 3y = 11 \end{cases} \Rightarrow \begin{cases} 15x + 12y = 18 \\ 16x - 12y = 44 \end{cases} \Rightarrow$
$31x = 62 \Rightarrow x = 2$
$5(2) + 4y = 6 \Rightarrow y = -1$

11. $\begin{cases} x - 3y + 6z = -8 \\ 5x - 6y - 2z = 7 \\ 3x - 2y - 10z = 11 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 1 & -3 & 6 & -8 \\ 5 & -6 & -2 & 7 \\ 3 & -2 & -10 & 11 \end{array}\right]$

$$\xrightarrow[R_3 - 3R_1 \to R_3]{R_2 - 5R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & -3 & 6 & -8 \\ 0 & 9 & -32 & 47 \\ 0 & 7 & -28 & 35 \end{array}\right]$$

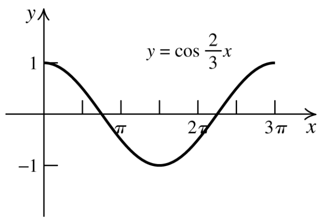
$$\xrightarrow{-\frac{1}{28}(9R_3 - 7R_2) \to R_3} \left[\begin{array}{ccc|c} 1 & -3 & 6 & -8 \\ 0 & 9 & -32 & 47 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right]$$

$$\xrightarrow{\frac{1}{9}(R_2 + 32R_3) \to R_2} \left[\begin{array}{ccc|c} 1 & -3 & 6 & -8 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right] \Rightarrow$$

$y = 7, z = \frac{1}{2}; x - 3(7) + 6\left(\frac{1}{2}\right) = -8 \Rightarrow x = 10$

The solution is $\left\{\left(10, 7, \frac{1}{2}\right)\right\}$.

12.



13. $\sin\left(-\frac{7\pi}{12}\right) = -\sin\left(\frac{7\pi}{12}\right) = -\sin\left(\frac{\pi}{4}+\frac{\pi}{3}\right)$

$= -\left(\sin\frac{\pi}{4}\cos\frac{\pi}{3} + \cos\frac{\pi}{4}\sin\frac{\pi}{3}\right)$

$= -\left(\frac{\sqrt{2}}{2}\cdot\frac{1}{2} + \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}\right)$

$= -\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right) = \frac{-\sqrt{2}-\sqrt{6}}{4}$

14. $r = \sqrt{2^2 + \left(2\sqrt{3}\right)^2} = 4$

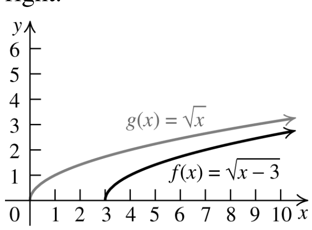
$\tan\theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$ or $\theta = \frac{4\pi}{3}$

$2 + 2\sqrt{3}i$ lies in Quadrant I, so $\theta = \frac{\pi}{3}$

The polar coordinates for $\left(2+2\sqrt{3}i\right)$ are

$4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$.

15. Shift the graph of $y = \sqrt{x}$ three units to the right.



16. The domain is $(-\infty,-1)\cup(-1,0)\cup(0,\infty)$.

17. $7\ln x - \ln(x-5) = \ln\left(\frac{x^7}{x-5}\right)$

18. $[A\,|\,I] = \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 4 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 7 & 0 & 0 & 1 \end{array}\right]$

$\xrightarrow[R_3 - R_1 \to R_3]{-4R_1 + R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 8 & -4 & 1 & 0 \\ 0 & 1 & 9 & -1 & 0 & 1 \end{array}\right]$

$\xrightarrow{R_3 - R_2 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 8 & -4 & 1 & 0 \\ 0 & 0 & 1 & 3 & -1 & 1 \end{array}\right]$

$\xrightarrow[R_2 - 8R_3 \to R_2]{R_1 + 2R_3 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -2 & 2 \\ 0 & 1 & 0 & -28 & 9 & -8 \\ 0 & 0 & 1 & 3 & -1 & 1 \end{array}\right] \Rightarrow$

$A^{-1} = \begin{bmatrix} 7 & -2 & 2 \\ -28 & 9 & -8 \\ 3 & -1 & 1 \end{bmatrix}$

19. Since order is important, find the number of permutations of four people taken two at a time: $P(4,2) = \frac{4!}{(4-2)!} = 12$.

There are 12 arrangements.

20. There are 18 people in total. Three people who are 25 years or older have received speeding tickets, and six people whose ages are between 17 and 24 have received a speeding ticket. The probability that a person chosen at random will be 25 years old or older is $\frac{6}{18}$. The probability that a person chosen at random will have received a speeding ticket is $\frac{9}{18}$. The probability that a person 25 years old or older will have received a speeding ticket is $\frac{3}{18}$. So, the probability that a person chosen at random from the room will be a person who is 25 years or older or has gotten a speeding ticket is $\frac{6}{18} + \frac{9}{18} - \frac{3}{18} = \frac{12}{18} = \frac{2}{3}$.