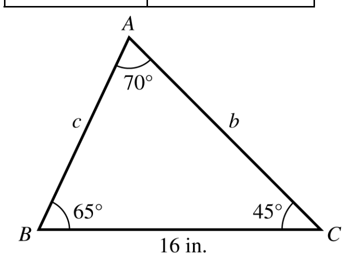
Chapter 7 Applications of Trigonometric Functions

7.1 The Law of Sines

7.1 Practice Problems

1. Given: $A = 70°$, $B = 65°$, and $a = 16$ in. – an AAS case.
Step 1: Find the third angle.
$C = 180° - (70° + 65°) = 45°$
Step 2: Make a chart.

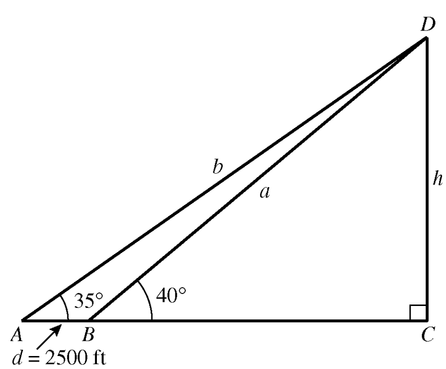
$A = 70°$	$a = 16$
$B = 65°$	$b = ?$
$C = 45°$	$c = ?$



Step 3: Apply the Law of Sines
$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{16}{\sin 70°} = \frac{b}{\sin 65°} \Rightarrow$$
$b \approx 15.4$ in.
$$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \frac{16}{\sin 70°} = \frac{c}{\sin 45°} \Rightarrow$$
$c \approx 12.0$ in.
Step 4: Show the solution.

$A = 70°$	$a = 16$ in.
$B = 65°$	**$b \approx 15.4$ in.**
$C = 45°$	**$c \approx 12.0$ in.**

2.



In triangle ABD, $m\angle DBC = 40° \Rightarrow$ $m\angle DBA = 180° - 40° = 140°$.
Then, $m\angle BDA = 180° - (35° + 140°) = 5°$.
$$\frac{2500}{\sin 5°} = \frac{a}{\sin 35°} \Rightarrow$$
$$a = \frac{2500 \sin 35°}{\sin 5°} \approx 16,453 \text{ ft}$$
In triangle BCD,
$$\sin 40° = \frac{h}{a} \Rightarrow h = a \sin 40° \Rightarrow$$
$h \approx 16,453 \sin 40° \approx 10,576$ ft

3. $A = 35°$ (an acute angle), $b = 6.5$, $a = 3.5$
Since $a \le b$, no triangles are possible.

4. $A = 105°$ (an obtuse angle), $b = 12.5$, $a = 11.2$
Since $a \le b$, no triangles are possible.

5. Given: $C = 35°$, $b = 15$ ft, and $c = 12$ ft – an SSA case.
Step 1: Make a chart.

$A = ?$	$a = ?$
$B = ?$	$b = 15$
$C = 35°$	$c = 12$

$\theta = C$, opposite side = c, adjacent side = b
Step 2: Count the solutions. Since $\theta = C = 35°$ is an acute angle, use Table 7.1.
$$\text{Altitude} = (\text{adjacent side}) \sin C = 15 \sin 35° \approx 8.6036$$
We have altitude < opposite side < adjacent side, so there will be two solutions.
Step 3: Apply the Law of Sines
$$\frac{\sin C}{c} = \frac{\sin B}{b} \Rightarrow \frac{\sin 35°}{12} = \frac{\sin B}{15} \Rightarrow$$
$$\sin B = \frac{15 \sin 35°}{12} \approx 0.7170$$
Step 4: Find the second angles.
$B_1 \approx \sin^{-1}(0.7170) \approx 45.8°$
$B_2 \approx 180° - 45.8° = 134.2°$
Step 5: Find the third angle of the triangles.
$A_1 \approx 180° - (35° + 45.8°) = 99.2°$
$A_2 \approx 180° - (35° + 134.2°) = 10.8°$
Step 6: Use the Law of Sines to find the remaining sides.
$$\frac{a_1}{\sin A_1} = \frac{c}{\sin C} \Rightarrow \frac{a_1}{\sin 99.2°} = \frac{12}{\sin 35°} \Rightarrow$$
$a_1 \approx 20.7$ ft
$$\frac{a_2}{\sin A_2} = \frac{c}{\sin C} \Rightarrow \frac{a_2}{\sin 10.8°} = \frac{12}{\sin 35°} \Rightarrow$$
$a_2 \approx 3.9$ ft

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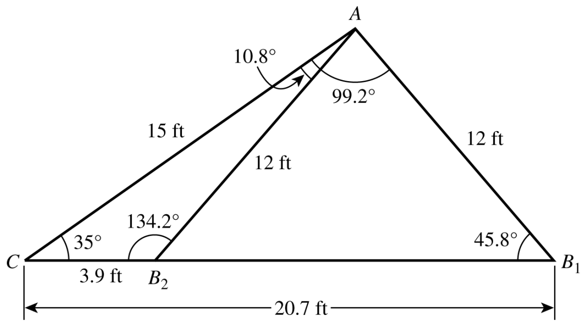
Step 7: Show the solutions.

$\triangle A_1BC: A_1 \approx 99.2°,\ B_1 \approx 45.8°,\ a_1 \approx 20.7$ ft

$\triangle A_2BC: A_2 \approx 10.8°,\ B_2 \approx 134.2°,\ a_2 \approx 3.9$ ft



6. Given: $A = 65°$, $a = 16$ m, and $b = 30$ m − an SSA case.

Step 1: Make a chart.

$A = 65°$	$a = 16$
$B = ?$	$b = 30$
$C = ?$	$c = ?$

Step 2: Count the solutions.

Since $\theta = A = 65°$ is an acute angle, use Table 7.1.

$$\text{Altitude} = (\text{adjacent side})\sin A = 30\sin 65° \approx 27.1892$$

We have opposite side < altitude, so there is no solution.

Alternatively, using the Law of Sines, we have

$$\frac{\sin A}{a} = \frac{\sin B}{b} \Rightarrow \frac{\sin 65°}{16} = \frac{\sin B}{30} \Rightarrow$$
$$\sin B \approx 1.6993$$

Since sin B cannot be greater than 1, no triangle is possible.

7. Given: $C = 60°$, $c = 50$ ft, and $a = 30$ ft − an SSA case.

Step 1: Make a chart.

$A = ?$	$a = 30$
$B = ?$	$b = ?$
$C = 60°$	$c = 50$

Step 2: Count the solutions.

Since $\theta = C = 60°$ is an acute angle, use Table 7.1. We have opposite side ≥ adjacent side, so there is one solution.

Step 3: Apply the Law of Sines.

$$\frac{\sin C}{c} = \frac{\sin A}{a} \Rightarrow \frac{\sin 60°}{50} = \frac{\sin A}{30} \Rightarrow$$
$$\sin A \approx 0.5196$$

Step 4: Find the second angle.

$$A = \sin^{-1}(0.5196) \approx 31.3°$$

Step 5: Find the third angle.

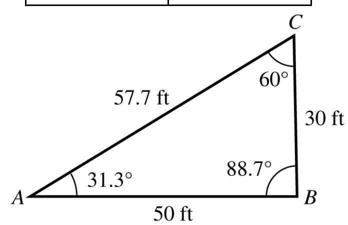
$$B = 180° - (31.3° + 60°) = 88.7°$$

Step 6: Use the Law of Sines to find the remaining side.

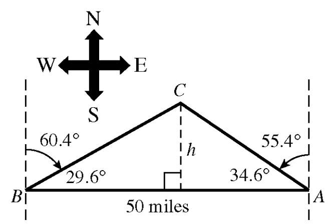
$$\frac{c}{\sin C} = \frac{b}{\sin B} \Rightarrow \frac{50}{\sin 60°} = \frac{b}{\sin 88.7°} \Rightarrow$$
$$b \approx 57.7 \text{ ft}$$

Step 7: Show the solution.

$\mathbf{A = 31.3°}$	$a = 30$ ft
$\mathbf{B = 88.7°}$	$\mathbf{b = 57.7}$ **ft**
$C = 60°$	c = 50 ft



8. a. In two hours, the ship travels (2)(25) = 50 mi.



The oil rig (at C), the starting point of the ship (at A) and the position of the ship after two hours (at B) are shown in the figure above. Since the original bearing is N 55.4º W, $m\angle CAB = 90° - 55.4° = 34.6°$. Since the second bearing is 60.4º,

$$m\angle CBA = 90° - 60.4° = 29.6°.$$
$$m\angle ACB = 180° - (29.6° + 34.6°) = 115.8°$$

Then $\dfrac{BC}{\sin \angle CAB} = \dfrac{AB}{\sin \angle ACB} \Rightarrow$

$$\frac{BC}{\sin 34.6°} = \frac{50}{\sin 115.8°} \Rightarrow BC \approx 31.5 \text{ mi}$$

The ship is approximately 31.5 mi from the oil rig when the second bearing is taken.

b. The shortest distance from the ship to the oil rig is the length of segment h.

$$\sin B = \frac{h}{BC} \Rightarrow \sin 29.6° \approx \frac{h}{31.5} \Rightarrow$$
$$h \approx 15.6 \text{ mi}$$

The ship passes within 15.6 miles of the oil rig.

7.1 Basic Concepts and Skills

1. If you know two angles of a triangle, then you can determine the third because the sum of all three angles is 180º degrees.

2. The Law of Sines states that if a, b, and c are the sides opposite angles A, B, and C, then $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

3. If you are given any two angles and one side, then there is exactly one triangle possible.

4. For given data a, A, and b, there can be no triangle, exactly one triangle, or two triangles.

5. True

6. False. A triangle is uniquely determined if any two sides and the included angle are given.

7. $a = 40, b = 70, A = 30°$
$h = b \sin A = 70 \sin 30° = 35$
Since $h < a < b$, there are two triangles with the given measurements.

8. $a = 24, b = 32, A = 45°$
$h = b \sin A = 32 \sin 45° \approx 22.6$
Since $h < a < b$, there are two triangles with the given measurements.

9. $b = 15, c = 19, B = 60°$
$h = c \sin B = 19 \sin 60° \approx 16.4545$
Since $h > b$, there are no triangles with the given measures.

10. $b = 75, c = 85, B = 135°$
Since B is obtuse and $b < c$, there are no triangles with the given measures.

11. $a = 50, b = 70, B = 120°$
Since B is obtuse and $b > a$, there is one triangle with the given measures.

12. $c = 85, a = 45, C = 150°$
Since C is obtuse and $c > a$, there is one triangle with the given measures.

13. $\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{2}{2/3} = \frac{3}{\sin B} \Rightarrow \sin B = 1 \Rightarrow$
$B = 90°$

14. $\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{\sqrt{6}}{\sin 60°} = \frac{2}{\sin C} \Rightarrow$
$\frac{\sqrt{6}}{\sqrt{3}/2} = \frac{2}{\sin C} \Rightarrow \sin C = \frac{\sqrt{2}}{2} \Rightarrow C = 45°$

15. $A + B + C = 180° \Rightarrow$
$A + \frac{1}{2}(A + C) + C = 180° \Rightarrow$
$\frac{3}{2}(A + C) = 180° \Rightarrow A + C = 120° \Rightarrow B = 60°$
$\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{\sqrt{3}}{\sin 60°} = \frac{\sqrt{2}}{\sin C} \Rightarrow$
$\frac{\sqrt{3}}{\sqrt{3}/2} = \frac{\sqrt{2}}{\sin C} \Rightarrow \sin C = \frac{\sqrt{2}}{2} \Rightarrow C = 45°$
$A = 180° - (60° + 45°) = 75°$

16. $A + B + C = 180° \Rightarrow$
$\frac{1}{3}(B + C) + B + C = 180° \Rightarrow$
$\frac{4}{3}(B + C) = 180° \Rightarrow B + C = 135° \Rightarrow A = 45°$
$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{4}{\sin 45°} = \frac{\sqrt{8}}{\sin B} \Rightarrow$
$\frac{4}{\sqrt{2}/2} = \frac{\sqrt{8}}{\sin B} \Rightarrow \sin B = \frac{1}{2} \Rightarrow B = 30°$

17. Given: $B = 45°$, $C = 105°$, and $a = 2$.
Find the third angle.
$A = 180° - (45° + 105)° = 30°$
Apply the Law of Sines
$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{2}{\sin 30°} = \frac{b}{\sin 45°} \Rightarrow$
$\frac{2}{1/2} = \frac{b}{\sqrt{2}/2} \Rightarrow 2\sqrt{2} = b$

18. Given: $A = 45°$, $B = 60°$, and $a = 10$.
Apply the Law of Sines
$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{10}{\sin 45°} = \frac{b}{\sin 60°} \Rightarrow$
$\frac{10}{\sqrt{2}/2} = \frac{b}{\sqrt{3}/2} \Rightarrow 5\sqrt{6} = b$

19. Given: $B = 45°$, $C = 75°$, and $b = \sqrt{6}$.
Find the third angle.
$A = 180° - (45° + 75)° = 60°$
Apply the Law of Sines
$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{a}{\sin 60°} = \frac{\sqrt{6}}{\sin 45°} \Rightarrow$
$\frac{a}{\sqrt{3}/2} = \frac{\sqrt{6}}{\sqrt{2}/2} \Rightarrow a = 3$

20. Given: $A = 105°$, $C = 45°$, and $b = \sqrt{12}$.
Find the third angle.
$B = 180° - (105° + 45)° = 30°$
Apply the Law of Sines
$$\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{\sqrt{12}}{\sin 30°} = \frac{c}{\sin 45°} \Rightarrow$$
$$\frac{\sqrt{12}}{1/2} = \frac{c}{\sqrt{2}/2} \Rightarrow c = 2\sqrt{6}$$

21. Given: $A = 61°, B = 56°, c = 100$ ft – an ASA case. $C = 180° - (61° + 56°) = 63°$
$$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \frac{a}{\sin 61°} = \frac{100}{\sin 63°} \Rightarrow$$
$a \approx 98.2$ ft
$$\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{b}{\sin 56°} = \frac{100}{\sin 63°} \Rightarrow$$
$b \approx 93.0$ ft

22. Given: $A = 40°, B = 35°, a = 60$ m – an AAS case. $C = 180° - (40° + 35°) = 105°$
$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{60}{\sin 40°} = \frac{b}{\sin 35°} \Rightarrow$$
$$b = \frac{60 \sin 35°}{\sin 40°} \approx 53.5 \text{ m}$$
$$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \frac{60}{\sin 40°} = \frac{c}{\sin 105°} \Rightarrow$$
$$c = \frac{60 \sin 105°}{\sin 40°} \approx 90.2 \text{ m}$$

23. Given: $A = 110°, C = 43°, c = 47$ m – an AAS case. $B = 180° - (43° + 110°) = 27°$
$$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \frac{a}{\sin 110°} = \frac{47}{\sin 43°} \Rightarrow$$
$$a = \frac{47 \sin 110°}{\sin 43°} \approx 64.8 \text{ m}$$
$$\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{b}{\sin 27°} = \frac{47}{\sin 43°} \Rightarrow$$
$$b = \frac{47 \sin 27°}{\sin 43°} \approx 31.3 \text{ m}$$

24. Given: $A = 46°, C = 93°, b = 22$ ft – an ASA case. $B = 180° - (46° + 93°) = 41°$
$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{a}{\sin 46°} = \frac{22}{\sin 41°} \Rightarrow$$
$$a = \frac{22 \sin 46°}{\sin 41°} \approx 24.1 \text{ m}$$
$$\frac{c}{\sin C} = \frac{b}{\sin B} \Rightarrow \frac{c}{\sin 93°} = \frac{22}{\sin 41°} \Rightarrow$$
$$c = \frac{22 \sin 93°}{\sin 41°} \approx 33.5 \text{ m}$$

25. Given: $A = 65°, a = 20$ ft, $b = 20$ ft – an SSA case. However, the triangle is isosceles, so $m\angle B = m\angle A = 65°$
$C = 180° - (65° + 65°) = 50°$
$$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \frac{20}{\sin 65°} = \frac{c}{\sin 50°} \Rightarrow c \approx 16.9 \text{ ft}$$

26. Given: $B = 40°, a = 18$ m, $b = 20$ m – an SSA case.
$$\frac{\sin B}{b} = \frac{\sin A}{a} \Rightarrow \frac{\sin 40°}{20} = \frac{\sin A}{18} \Rightarrow$$
$\sin A \approx 0.5785 \Rightarrow A_1 \approx 35.3°$ or $A_2 \approx 144.7°$
$C_1 = 180° - (40° + 35.3°) = 104.7°$
$C_2 = 180° - (40° + 144.7°) = -4.7°$, so there is one solution, with $A = 35.3°$ and $C = 104.7°$.
$$\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{20}{\sin 40°} = \frac{c}{\sin 104.7°} \Rightarrow$$
$c \approx 30.1$ m

27. Given: $A = 115°, a = 70$ m, $b = 31$ m – an SSA case.
$$\frac{\sin A}{a} = \frac{\sin B}{b} \Rightarrow \frac{\sin 115°}{70} = \frac{\sin B}{31} \Rightarrow$$
$\sin B \approx 0.4014 \Rightarrow B_1 \approx 23.7°$ or $B_2 \approx 156.3°$
$C_1 = 180° - (115° + 23.7°) = 41.3°$
$C_2 = 180° - (115° + 156.3°) = -91.3°$, so there is one solution, with $B = 23.7°$ and $C = 41.3°$.
$$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \frac{70}{\sin 115°} = \frac{c}{\sin 41.3°} \Rightarrow$$
$c \approx 51.0$ m

28. Given: $C = 115°, a = 12$ ft, $c = 14$ ft – an SSA case.
$$\frac{\sin C}{c} = \frac{\sin A}{a} \Rightarrow \frac{\sin 115°}{14} = \frac{\sin A}{12} \Rightarrow$$
$\sin A \approx 0.7768 \Rightarrow A_1 \approx 51.0°$ or $A_2 \approx 129.0°$
$B_1 = 180° - (115° + 51.0°) = 14.0°$
$B_2 = 180° - (115° + 129.0°) = -64°$, so there is one solution, with $A = 51.0°$ and $B = 14.0°$.
$$\frac{c}{\sin C} = \frac{b}{\sin B} \Rightarrow \frac{14}{\sin 115°} = \frac{b}{\sin 14.0°} \Rightarrow$$
$b \approx 3.7$ ft

29. Given: $A = 40°, B = 35°, a = 100$ m – an AAS case. $C = 180° - (40° + 35°) = 105°$

$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{100}{\sin 40°} = \frac{b}{\sin 35°} \Rightarrow$
$b \approx 89.2$ m

$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \frac{100}{\sin 40°} = \frac{c}{\sin 105°} \Rightarrow$
$c \approx 150.3$ m

30. Given: $A = 80°, B = 20°, a = 100$ m – an AAS case. $C = 180° - (80° + 20°) = 80°$

$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{100}{\sin 80°} = \frac{b}{\sin 20°} \Rightarrow b \approx 34.7$ m
$m\angle A = m\angle C = 80° \Rightarrow \triangle ABC$ is isosceles, so $c = 100$ m

31. Given: $A = 46°, C = 55°, a = 75$ cm – an AAS case. $B = 180° - (46° + 55°) = 79°$

$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{75}{\sin 46°} = \frac{b}{\sin 79°} \Rightarrow$
$b \approx 102.3$ cm

$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \frac{75}{\sin 46°} = \frac{c}{\sin 55°} \Rightarrow$
$c \approx 85.4$ cm

32. Given: $A = 35°, C = 98°, a = 75$ cm – an AAS case. $B = 180° - (35° + 98°) = 47°$

$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{75}{\sin 35°} = \frac{b}{\sin 47°} \Rightarrow$
$b \approx 95.6$ cm

$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \frac{75}{\sin 35°} = \frac{c}{\sin 98°} \Rightarrow$
$c \approx 129.5$ cm

33. Given: $A = 35°, C = 47°, c = 60$ ft – an AAS case. $B = 180° - (35° + 47°) = 98°$

$\frac{c}{\sin C} = \frac{a}{\sin A} \Rightarrow \frac{60}{\sin 47°} = \frac{a}{\sin 35°} \Rightarrow$
$a \approx 47.1$ ft

$\frac{c}{\sin C} = \frac{b}{\sin B} \Rightarrow \frac{60}{\sin 47°} = \frac{b}{\sin 98°} \Rightarrow$
$c \approx 81.2$ ft

34. Given: $A = 44°, C = 76°, c = 40$ ft – an AAS case. $B = 180° - (44° + 76°) = 60°$

$\frac{c}{\sin C} = \frac{a}{\sin A} \Rightarrow \frac{40}{\sin 76°} = \frac{a}{\sin 44°} \Rightarrow$
$a \approx 28.6$ ft

$\frac{c}{\sin C} = \frac{b}{\sin B} \Rightarrow \frac{40}{\sin 76°} = \frac{b}{\sin 60°} \Rightarrow$
$b \approx 35.7$ ft

35. Given: $B = 43°, C = 67°, b = 40$ in. – an AAS case. $A = 180° - (43° + 67°) = 70°$

$\frac{b}{\sin B} = \frac{a}{\sin A} \Rightarrow \frac{40}{\sin 43°} = \frac{a}{\sin 70°} \Rightarrow$
$a \approx 55.1$ in.

$\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{40}{\sin 43°} = \frac{c}{\sin 67°} \Rightarrow$
$c \approx 54.0$ in.

36. Given: $B = 95°, C = 35°, b = 100$ in. – an AAS case. $A = 180° - (95° + 35°) = 50°$

$\frac{b}{\sin B} = \frac{a}{\sin A} \Rightarrow \frac{100}{\sin 95°} = \frac{a}{\sin 50°} \Rightarrow$
$a \approx 76.9$ in.

$\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{100}{\sin 95°} = \frac{c}{\sin 35°} \Rightarrow$
$c \approx 57.6$ in.

37. Given: $B = 110°, C = 46°, c = 23.5$ ft – an AAS case. $A = 180° - (110° + 46°) = 24°$

$\frac{c}{\sin C} = \frac{a}{\sin A} \Rightarrow \frac{23.5}{\sin 46°} = \frac{a}{\sin 24°} \Rightarrow$
$a \approx 13.3$ ft

$\frac{c}{\sin C} = \frac{b}{\sin B} \Rightarrow \frac{23.5}{\sin 46°} = \frac{b}{\sin 110°} \Rightarrow$
$b \approx 30.7$ ft

38. Given: $B = 67°, C = 63°, c = 16.8$ ft – an AAS case. $A = 180° - (67° + 63°) = 50°$

$\frac{c}{\sin C} = \frac{a}{\sin A} \Rightarrow \frac{16.8}{\sin 63°} = \frac{a}{\sin 50°} \Rightarrow$
$a \approx 14.4$ ft

$\frac{c}{\sin C} = \frac{b}{\sin B} \Rightarrow \frac{16.8}{\sin 63°} = \frac{b}{\sin 67°} \Rightarrow$
$b \approx 17.4$ ft

39. Given: $A = 35.7°, B = 45.8°, c = 30$ m – an ASA case. $C = 180° - (35.7° + 45.8°) = 98.5°$

$\frac{c}{\sin C} = \frac{a}{\sin A} \Rightarrow \frac{30}{\sin 98.5°} = \frac{a}{\sin 35.7°} \Rightarrow$
$a \approx 17.7$ m

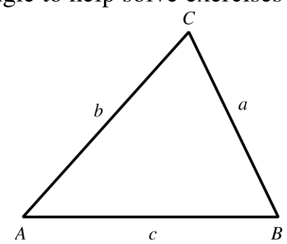
$\frac{c}{\sin C} = \frac{b}{\sin B} \Rightarrow \frac{30}{\sin 98.5°} = \frac{b}{\sin 45.8°} \Rightarrow$
$b \approx 21.7$ m

40. Given: $A = 64.5°, B = 54.3°, c = 40$ m – an ASA case. $C = 180° - (64.5° + 54.3°) = 61.2°$

$\frac{c}{\sin C} = \frac{a}{\sin A} \Rightarrow \frac{40}{\sin 61.2°} = \frac{a}{\sin 64.5°} \Rightarrow$
$a \approx 41.2$ m

$\frac{c}{\sin C} = \frac{b}{\sin B} \Rightarrow \frac{40}{\sin 61.2°} = \frac{b}{\sin 54.3°} \Rightarrow$
$b \approx 37.1$ m

Use this triangle to help solve exercises 41−60.



41. Given: $A = 40°, a = 23, b = 20$.
$h = 20\sin 40° \approx 12.9$; $h < b < a$, so there is one triangle.
$$\frac{\sin A}{a} = \frac{\sin B}{b} \Rightarrow \frac{\sin 40°}{23} = \frac{\sin B}{20} \Rightarrow$$
$$B \approx \sin^{-1}\left(\frac{20\sin 40°}{23}\right) \approx 34.0°$$
$$C = 180° - (40° + 34.0°) \approx 106.0°$$
$$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \frac{23}{\sin 40°} = \frac{c}{\sin 106.0°} \Rightarrow c \approx 34.4$$

42. Given: $A = 36°, a = 30, b = 24$;
$h = 24\sin 36° \approx 14.1$; $h < b < a$, so there is one triangle.
$$\frac{\sin A}{a} = \frac{\sin B}{b} \Rightarrow \frac{\sin 36°}{30} = \frac{\sin B}{24} \Rightarrow$$
$$B \approx \sin^{-1}\left(\frac{24\sin 36°}{30}\right) \approx 28.0°$$
$$C = 180° - (36° + 28.0°) \approx 116.0°$$
$$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \frac{30}{\sin 36°} = \frac{c}{\sin 116.0°} \Rightarrow$$
$c \approx 45.9$

43. Given: $A = 30°, a = 25, b = 50$;
$h = b\sin A = 50\sin 30° = 25.$ $a = h$, so there is one right triangle.
$B = 90°, C = 60°, c = 25\sqrt{3}$ (using the Pythagorean theorem).

44. Given: $A = 60°, a = 20\sqrt{3}, b = 40$;
$h = b\sin A = 40\sin 60° = 20\sqrt{3}.$ $a = h$, so there is one right triangle.
$B = 90°, C = 30°, c = 20$ (using the Pythagorean theorem).

45. Given: $A = 40°, a = 10, b = 20$. A is an acute angle. $h = b\sin A = 20\sin 40° \approx 12.9$; $a < h$, so no triangle exists.

46. Given: $A = 62°, a = 30, b = 40$. A is an acute angle. $h = b\sin A = 40\sin 62° \approx 35.3$; $a < h$, so no triangle exists.

47. Given: $A = 95°, a = 18, b = 21$. A is an obtuse angle and $a < b$, so no triangle exists.

48. Given: $A = 110°, a = 37, b = 41$. A is an obtuse angle and $a < b$, so no triangle exists.

49. Given: $A = 100°, a = 40, b = 34$. $b < a$, so there is one triangle.
$$\frac{\sin A}{a} = \frac{\sin B}{b} \Rightarrow \frac{\sin 100°}{40} = \frac{\sin B}{34} \Rightarrow$$
$$B \approx \sin^{-1}\left(\frac{34\sin 100°}{40}\right) \approx 56.8°$$
$$C = 180° - (100° + 56.8°) \approx 23.2°$$
$$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \frac{40}{\sin 100°} = \frac{c}{\sin 23.2°} \Rightarrow$$
$c \approx 16.0$

50. Given: $A = 105°, a = 70, b = 30$. $b < a$, so there is one triangle.
$$\frac{\sin A}{a} = \frac{\sin B}{b} \Rightarrow \frac{\sin 105°}{70} = \frac{\sin B}{30} \Rightarrow$$
$$B \approx \sin^{-1}\left(\frac{30\sin 105°}{70}\right) \Rightarrow B \approx 24.45° \approx 24.5°$$
$$C = 180° - (105° + 24.45°) \approx 50.55°$$
$$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \frac{70}{\sin 105°} = \frac{c}{\sin 50.55°} \Rightarrow$$
$c \approx 56.0$

51. Given: $B = 50°, b = 22, c = 40$. B is an acute angle. $h = c\sin B = 40\sin 50° \approx 30.6$; $b < h$, so no triangle exists.

52. Given: $B = 64°, b = 45, c = 60$. B is an acute angle. $h = c\sin B = 60\sin 64° \approx 53.9$; $b < h$, so no triangle exists.

53. Given: $B = 46°, b = 35, c = 40$.
$h = c\sin B = 40\sin 46° \approx 28.8$; $h < b < c$, so two triangles exist. $\frac{\sin B}{b} = \frac{\sin C}{c} \Rightarrow$
$$\frac{\sin 46°}{35} = \frac{\sin C}{40} \Rightarrow C = \sin^{-1}\left(\frac{40\sin 46°}{35}\right) \Rightarrow$$
$C_1 \approx 55.3°, C_2 \approx 124.7°$
$A_1 = 180° - (46° + 55.3°) \approx 78.7°$
$$\frac{b}{\sin B} = \frac{a_1}{\sin A_1} \Rightarrow \frac{35}{\sin 46°} = \frac{a_1}{\sin 78.7°} \Rightarrow$$
$a_1 \approx 47.7$
$A_2 = 180° - (46° + 124.7°) \approx 9.3°$
$$\frac{b}{\sin B} = \frac{a_2}{\sin A_2} \Rightarrow \frac{35}{\sin 46°} = \frac{a_2}{\sin 9.3°} \Rightarrow$$
$a_2 \approx 7.9$. The two solutions are: $C_1 \approx 55.3°$, $A_1 \approx 78.7°, a_1 \approx 47.7$ and $C_2 \approx 124.7°$, $A_2 \approx 9.3°, a_2 \approx 7.9$.

54. Given: $B = 32°, b = 50, c = 60$.
$h = c \sin B = 60 \sin 32° \approx 31.8$; $h < b < c$, so two triangles exist. $\frac{\sin B}{b} = \frac{\sin C}{c} \Rightarrow$
$\frac{\sin 32°}{50} = \frac{\sin C}{60} \Rightarrow C = \sin^{-1}\left(\frac{60 \sin 32°}{50}\right) \Rightarrow$
$C_1 \approx 39.5°, C_2 \approx 140.5°$
$A_1 = 180° - (32° + 39.5°) \approx 108.5°$
$\frac{b}{\sin B} = \frac{a_1}{\sin A_1} \Rightarrow \frac{50}{\sin 32°} = \frac{a_1}{\sin 108.5°} \Rightarrow$
$a_1 \approx 89.5$
$A_2 = 180° - (32° + 140.5°) \approx 7.5°$
$\frac{b}{\sin B} = \frac{a_2}{\sin A_2} \Rightarrow \frac{50}{\sin 32°} = \frac{a_2}{\sin 7.5°} \Rightarrow$
$a_2 \approx 12.3$. The two solutions are: $C_1 \approx 39.5°$, $A_1 \approx 108.5°, a_1 \approx 89.5$ and $C_2 \approx 140.5°$, $A_2 \approx 7.5°, a_2 \approx 12.3$.

55. Given: $B = 97°, b = 27, c = 30$. B is an obtuse angle and $b < c$, so no triangle exists.

56. Given: $B = 110°, b = 19, c = 21$. B is an obtuse angle and $b < c$, so no triangle exists.

57. Given: $A = 42°, a = 55, c = 62$.
$h = c \sin A = 62 \sin 42° \approx 41.5$; $h < a < c$, so two triangles exist. $\frac{\sin A}{a} = \frac{\sin C}{c} \Rightarrow$
$\frac{\sin 42°}{55} = \frac{\sin C}{62} \Rightarrow C = \sin^{-1}\left(\frac{62 \sin 42°}{55}\right) \Rightarrow$
$C_1 \approx 49.0°, C_2 \approx 131.0°$
$B_1 = 180° - (42° + 49.0°) \approx 89.0°$
$\frac{a}{\sin A} = \frac{b_1}{\sin B_1} \Rightarrow \frac{55}{\sin 42°} = \frac{b_1}{\sin 89.0°} \Rightarrow$
$b_1 \approx 82.2$
$B_2 = 180° - (42° + 131.0°) \approx 7.0°$
$\frac{a}{\sin A} = \frac{b_2}{\sin B_2} \Rightarrow \frac{55}{\sin 42°} = \frac{b_2}{\sin 7.0°} \Rightarrow$
$b_2 \approx 10.0$.
The two solutions are: $C_1 \approx 49.0°$, $B_1 \approx 89.0°, b_1 \approx 82.2$ and $C_2 \approx 131.0°$, $B_2 \approx 7.0°, b_2 \approx 10.0$.

58. Given: $A = 34°, a = 6, c = 8$.
$h = c \sin A = 8 \sin 34° \approx 4.5$
$h < a < c$, so two triangles exist.
$\frac{\sin A}{a} = \frac{\sin C}{c} \Rightarrow$
$\frac{\sin 34°}{6} = \frac{\sin C}{8} \Rightarrow C = \sin^{-1}\left(\frac{8 \sin 34°}{6}\right) \Rightarrow$
$C_1 \approx 48.2°, C_2 \approx 131.8°$
$B_1 = 180° - (34° + 48.2°) \approx 97.8°$
$\frac{a}{\sin A} = \frac{b_1}{\sin B_1} \Rightarrow \frac{6}{\sin 34°} = \frac{b_1}{\sin 97.8°} \Rightarrow$
$b_1 \approx 10.6$
$B_2 = 180° - (34° + 131.8°) \approx 14.2°$
$\frac{a}{\sin A} = \frac{b_2}{\sin B_2} \Rightarrow \frac{6}{\sin 34°} = \frac{b_2}{\sin 14.2°} \Rightarrow$
$b_2 \approx 2.6$.
The two solutions are: $C_1 \approx 48.2°$, $B_1 \approx 97.8°, b_1 \approx 10.6$ and $C_2 \approx 131.8°$, $B_2 \approx 14.2°, b_2 \approx 2.6$.

59. Given: $C = 40°, a = 3.3, c = 2.1$.
$h = 3.3 \sin 40° \approx 2.1212 > c$, so no triangle exists.

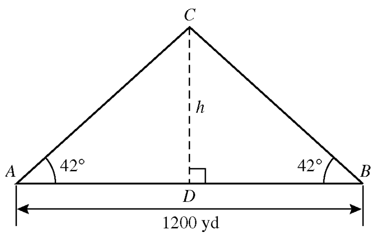
60. Given: $C = 62°, a = 50, c = 100$. C is acute and $c > a$, so one triangle exists. $\frac{\sin C}{c} = \frac{\sin A}{a} \Rightarrow$
$\frac{\sin 62°}{100} = \frac{\sin A}{50} \Rightarrow$
$A = \sin^{-1}\left(\frac{50 \sin 62°}{100}\right) \approx 26.2°$
$B = 180° - (62° + 26.2°) \approx 91.8°$
$\frac{c}{\sin C} = \frac{b}{\sin B} \Rightarrow \frac{100}{\sin 62°} = \frac{b}{\sin 91.8°} \Rightarrow$
$b \approx 113.2$

7.1 Applying the Concepts

61. The length of the side from A to C is b.
$C = 180° - (57° + 46°) = 77°$
$\frac{c}{\sin C} = \frac{b}{\sin B} \Rightarrow \frac{540}{\sin 77°} = \frac{b}{\sin 46°} \Rightarrow$
$b \approx 399$ ft

62. The width of the river is the altitude drawn from C to side AB:
$h = b \sin 57° = \frac{540 \sin 46°}{\sin 77°} \sin 57° \approx 334$ ft.

63. a.



The total distance the laser beam travels $= AC + BC = 2AC$.

$C = 180° - 2(42°) = 96°$.

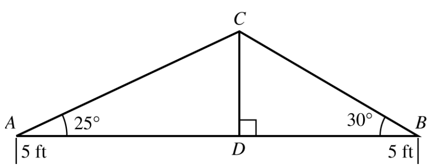
$$\frac{1200}{\sin 96°} = \frac{AC}{\sin 42°} \Rightarrow AC \approx 807.4.$$

The total distance the laser beam traveled is approximately 1615 yards.

b. The height of the target is the altitude drawn from *C* to side *AB*.

$h = 807.4 \sin 42° \approx 540$ yd

64. A transit is a surveying instrument that measures horizontal and vertical angles.

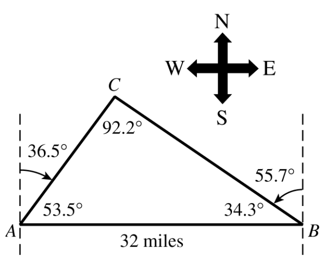


$C = 180° - (25° + 30°) = 125°$. $c = AB = 200$;

$$\frac{200}{\sin 125°} = \frac{AC}{\sin 30°} \Rightarrow AC \approx 122 \text{ ft}$$

$CD = 122 \sin 25° \approx 52$ ft. The height of the flagpole is $52 + 5 \approx 57$ ft.

65. a.



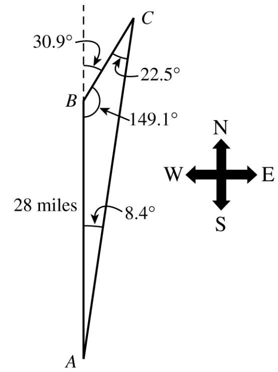
At 16 miles per hour, the ship travels 32 miles in two hours. The interior angles of the triangle are shown in the figure above. The distance from the ship at the second time to the beacon is *BC*.

$$\frac{32}{\sin 92.2°} = \frac{BC}{\sin 53.5°} \Rightarrow BC \approx 25.7 \text{ mi}$$

b. The closest the ship came to the beacon is the length of the altitude from *C* to *AB*:

$h = 25.7 \sin 34.3° \approx 14.5$ mi

66. a.

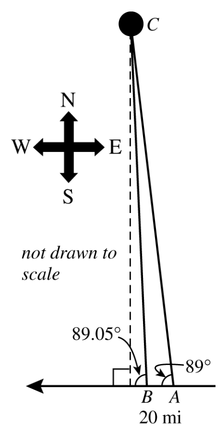


At 14 miles per hour, the ship travels 28 miles in two hours. The lighthouse is located at C. The interior angles of the triangle are shown in the figure. The distance from the boat at the second time to the lighthouse is *BC*.

$$\frac{28}{\sin 22.5°} = \frac{BC}{\sin 8.4°} \Rightarrow BC \approx 10.7 \text{ mi}$$

b. The closest the boat comes to the lighthouse is the length of the altitude from *C* to *AB*: $h = 10.7 \sin 30.9° \approx 5.5$ mi

67.



$m\angle CBA = 180° - 89.05° = 90.95°$

$C = 180° - (90.95° + 89°) = 0.05°$

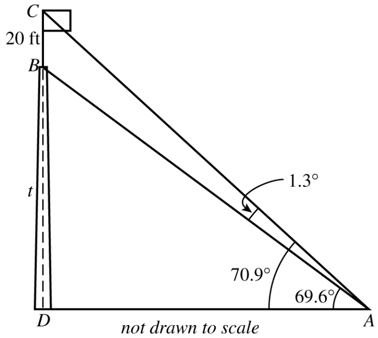
$$\frac{AC}{\sin 90.95°} = \frac{20}{\sin 0.05°} \Rightarrow AC = \frac{20 \sin 90.95°}{\sin 0.05°}$$

The height of the satellite is the length of the altitude drawn from *C* to *AB*:

$$h = AC \sin 89° = \left(\frac{20 \sin 90.95°}{\sin 0.05°}\right) \sin 89°$$

$$\approx 22{,}912 \text{ mi}$$

68.



$m\angle BAC = 70.9° - 69.6° = 1.3°$

In $\triangle ABD, m\angle ABD = 90° - 69.6° = 20.4°$, so $m\angle ABC = 180° - 20.4° = 159.6°$. Therefore, $m\angle BCA = 180° - (159.6° + 1.3°) = 19.1°$.

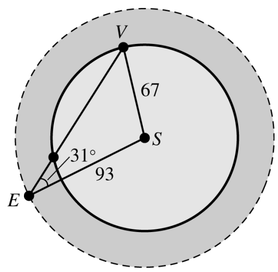
$$\frac{AB}{\sin\angle BCA} = \frac{20}{\sin 1.3°} \Rightarrow AB = \frac{20\sin 19.1°}{\sin 1.3°}$$

$$\frac{t}{\sin 69.6°} = \frac{AB}{\sin 90°} \Rightarrow$$

$$t = \sin 69.6°\left(\frac{20\sin 19.1°}{\sin 1.3°}\right) \approx 270 \text{ ft}$$

In exercises 69 and 70, be sure to carry all decimal places throughout the exercise.

69.



$h = 93 \sin 31° \approx 47.9$

Since $h < VS < ES$, there are two possible solutions.

$$\frac{\sin E}{VS} = \frac{\sin V}{ES} \Rightarrow \frac{\sin 31°}{67} = \frac{\sin V}{93} \Rightarrow$$

$$V_1 = \sin^{-1}\left(\frac{93\sin 31°}{67}\right) \approx 45.6° \text{ and}$$

$$V_2 \approx 180° - 45.6352991° \approx 134.3647009°$$

$$S_1 = 180° - (31° + 45.6352991°) = 103.3647009°$$

$$S_2 = 180° - (31° + 134.3647009°) = 14.6352991°$$

$$\frac{\sin S_1}{EV_1} = \frac{\sin E}{VS} \Rightarrow \frac{\sin 103.3647009°}{EV_1} = \frac{\sin 31°}{67} \Rightarrow$$

$$EV_1 = \frac{67\sin 103.3647009°}{\sin 31°} \approx 126.5645021$$

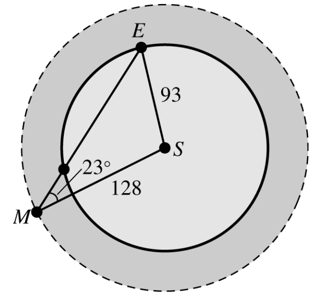
$$\frac{\sin S_2}{EV_2} = \frac{\sin E}{VS} \Rightarrow$$

$$\frac{\sin 14.6352991°°}{EV_2} = \frac{\sin 31°}{67} \Rightarrow$$

$$EV_2 = \frac{67\sin 14.6352991°}{\sin 31°} \approx 32.86861586$$

The distance between Earth and Venus is approximately 127 million miles or 33 million miles.

70.



$h = 128 \sin 23° \approx 50.0$

Since $h < ES < MS$, there are two possible solutions.

$$\frac{\sin M}{ES} = \frac{\sin E}{MS} \Rightarrow \frac{\sin 23°}{93} = \frac{\sin E}{128} \Rightarrow$$

$$E_1 = \sin^{-1}\left(\frac{128\sin 23°}{93}\right) \approx 32.5326738° \text{ and}$$

$$E_2 \approx 180° - 32.5326738° \approx 147.4673262°.$$

$$S_1 = 180° - (23° + 32.5326738°) = 124.4673262°$$

$$S_2 = 180° - (23° + 147.4673262°) = 9.532673801°$$

$$\frac{\sin S_1}{EM_1} = \frac{\sin M}{ES} \Rightarrow$$

$$\frac{\sin 124.4673262°}{EM_1} = \frac{\sin 23°}{93} \Rightarrow$$

$$EM_1 = \frac{93\sin 116.4673262°}{\sin 23°} \approx 196.2$$

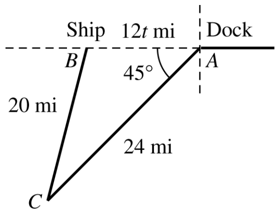
$$\frac{\sin S_2}{EM_2} = \frac{\sin M}{ES} \Rightarrow$$

$$\frac{\sin 9.532673801°}{EM_2} = \frac{\sin 23°}{93} \Rightarrow$$

$$EM_2 = \frac{93\sin 9.532673801°}{\sin 23°} \approx 39.4$$

The distance between Earth and Venus is approximately 196 million miles or 39 million miles.

71.



$h = 24 \sin 45° \approx 17$

Since $h < BC < AC$, there are two possible solutions.

$$\frac{\sin B}{AC} = \frac{\sin A}{BC} \Rightarrow \frac{\sin B}{24} = \frac{\sin 45°}{20} \Rightarrow$$

$$B_1 = \sin^{-1}\left(\frac{24\sin 45°}{20}\right) \approx 58.1°$$

$$B_2 = 180° - B_1 \approx 121.9°$$

$$C_1 = 180° - (45° + 58.1°) = 76.9°$$

$$C_2 = 180° - (45° + 121.9°) = 13.1°$$

$$\frac{AB_1}{\sin C_1} = \frac{BC}{\sin A} \Rightarrow$$

$$AB_1 = \frac{20\sin 76.9°}{\sin 45°} \approx 27.548 \text{ mi}$$

Since the ship traveled at 12 mph, it traveled

$$\frac{27.548}{12} = 2.296 \text{ hr} = 2 \text{ hr } 18 \text{ min.}$$

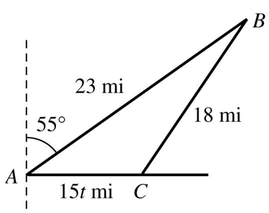
$$\frac{AB_2}{\sin C_2} = \frac{BC}{\sin A} \Rightarrow$$

$$AB_1 = \frac{20\sin 13.1°}{\sin 45°} \approx 6.411 \text{ mi}$$

$$\frac{6.411}{12} = 0.53 \text{ hr} = 32 \text{ min.}$$

The two times are 1:32 pm and 3:18 pm.

72.



The dock is located at A, the lighthouse at B, and the ship at C.

$m\angle BAC = 90° - 55° = 35°.$

$$\frac{\sin \angle BAC}{BC} = \frac{\sin \angle BCA}{AB} \Rightarrow$$

$$\frac{\sin 35°}{18} = \frac{\sin \angle BCA}{23} \Rightarrow$$

$$m\angle BCA_1 = \sin^{-1}\left(\frac{23\sin 35°}{18}\right) \approx 47.1°$$

$$m\angle BCA_2 = 180° - 47.1° = 132.9°$$

$$B_1 = 180° - (35° + 47.1°) = 97.9°$$

$$B_2 = 180° - (35° + 132.9°) = 12.1°$$

$$\frac{AC}{\sin B_1} = \frac{BC}{\sin \angle BAC} \Rightarrow \frac{AC}{\sin 97.9°} = \frac{18}{\sin 35°} \Rightarrow$$

$$AC = \frac{18\sin 97.9°}{\sin 35°} \approx 31.1 \text{ mi}$$

$15t = 31.1 \Rightarrow t \approx 2.072$ hr $\approx$ 2 hr 4 min

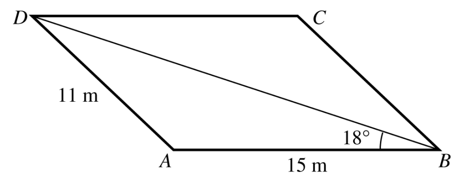
$$\frac{AC}{\sin B_2} = \frac{BC}{\sin \angle BAC} \Rightarrow \frac{AC}{\sin 12.1°} = \frac{18}{\sin 35°} \Rightarrow$$

$$AC = \frac{18\sin 12.1°}{\sin 35°} \approx 6.6 \text{ mi}$$

$15t = 6.6 \Rightarrow t \approx 0.44$ hr $\approx$ 26 min

The two times are 3:26 pm and 5:04 pm.

73.



$$\frac{\sin 18°}{11} = \frac{\sin \angle ADC}{15} \Rightarrow$$

$$\sin \angle ADC = \frac{15\sin 18°}{11} \Rightarrow$$

$$m\angle ADC = \sin^{-1}\left(\frac{15\sin 18°}{11}\right) \approx 24.9°$$

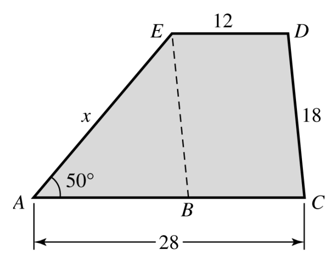
$$m\angle A = 180° - (18° + 24.9°) = 137.1°$$

$$\frac{BD}{\sin 137.1°} = \frac{11}{\sin 18°} \Rightarrow$$

$$BD = \frac{11\sin 137.1°}{\sin 18°} \approx 24.2$$

The diagonal is about 24.2 m.

74.



Construct EB parallel to DC. Then, $EB = DC = 18$, and $ED = BC = 12$. Therefore, $AB = 28 - 12 = 16$.

$$\frac{\sin A}{EB} = \frac{\sin \angle BEA}{AB} \Rightarrow \frac{\sin 50°}{18} = \frac{\sin \angle BEA}{16} \Rightarrow$$

$$\sin \angle BEA = \frac{16\sin 50°}{18} \Rightarrow$$

$$m\angle BEA = \sin^{-1}\left(\frac{16\sin 50°}{18}\right) \approx 42.9°$$

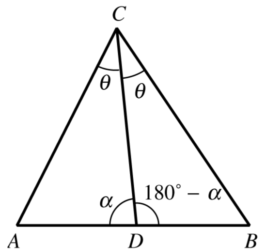
$$m\angle EBA = 180° - (42.9° + 50°) = 87.1°$$

$$\frac{EA}{\sin 87.1°} = \frac{18}{\sin 50°} \Rightarrow EA = \frac{18\sin 87.1°}{\sin 50°} \Rightarrow$$

$EA \approx 23.5$

7.1 Beyond the Basics

75.



In $\triangle ACD$, we have $\frac{AD}{\sin\theta} = \frac{AC}{\sin\alpha} \Rightarrow$

$\frac{AD}{AC} = \frac{\sin\theta}{\sin\alpha}$. In $\triangle DCB$, we have

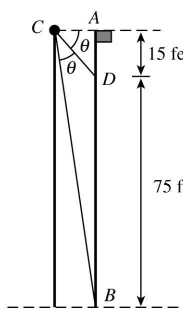
$\frac{BD}{\sin\theta} = \frac{CB}{\sin(180° - \alpha)} \Rightarrow \frac{BD}{CB} = \frac{\sin\theta}{\sin(180° - \alpha)}$

$\sin(180° - \alpha) = \sin\alpha$, so

$\frac{BD}{CB} = \frac{\sin\theta}{\sin(180° - \alpha)} = \frac{\sin\theta}{\sin\alpha} = \frac{AD}{AC} \Rightarrow$

$\frac{AD}{BC} = \frac{AC}{CB}$

76.



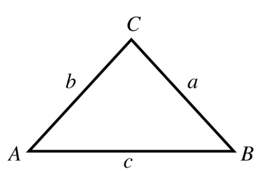
The observer is located at *C*. Using the result from exercise 71, we have

$\frac{AC}{CB} = \frac{AD}{BC} = \frac{15}{75} = \frac{1}{5}$ So, $\sin\angle ABC = \frac{1}{5}$ and

$\tan\angle ABC = \frac{CA}{90} \Rightarrow$

$CA = 90\tan\left(\sin^{-1}\frac{1}{5}\right) \approx 18.4$ ft

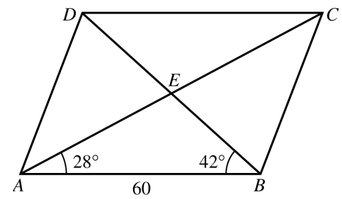
77.



Given: $a\cos A = b\cos B$. Using the Law of Sines, we have $\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow a = \frac{b\sin A}{\sin B}$.
Substituting in the given equation gives

$\frac{b\sin A}{\sin B} \cdot \cos A = b\cos B \Rightarrow$
$\sin A\cos A = \sin B\cos B \Rightarrow$
$2\sin A\cos A = 2\sin B\cos B \Rightarrow$
$\sin 2A = \sin 2B$
Then, $2A = 2B \Rightarrow A = B$ or
If $A = B$, then the triangle is isosceles.
If $A = 90° - B$, then the triangle is a right triangle.

78.



$m\angle AEB = 180° - (28° + 42°) = 110°$

$\frac{60}{\sin 110°} = \frac{AE}{\sin 42°} \Rightarrow AE \approx 42.7 \Rightarrow$
$AC = 2(42.7) \approx 85.4$ in.

$\frac{60}{\sin 110°} = \frac{BE}{\sin 28°} \Rightarrow BE \approx 30.0 \Rightarrow$
$BD = 2(30.0) \approx 60.0$ in.

The diagonals are 60.0 in. and 85.4 in.

79. Let $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$, so $a = k\sin A$, $B = k\sin B$, and $C = k\sin C$.

Then $\frac{b-c}{a} = \frac{k(\sin B - \sin C)}{k\sin A} = \frac{\sin B - \sin C}{\sin A}$.

Using the sum-to-product formula, this becomes

$$\frac{2\sin\left(\frac{B-C}{2}\right)\cos\left(\frac{B+C}{2}\right)}{\sin A}$$

$$. \frac{(A+B+C)}{2} = 90° \Rightarrow$$

$\frac{B+C}{2} = 90° - \frac{A}{2}$, so

$\cos\left(\frac{B+C}{2}\right) = \cos\left(90° - \frac{A}{2}\right) = \sin\frac{A}{2}$.

Substituting and using the double angle formula in the denominator, we have

$$\frac{2\sin\left(\frac{B-C}{2}\right)\cos\left(\frac{B+C}{2}\right)}{\sin A} = \frac{2\sin\left(\frac{B-C}{2}\right)\sin\frac{A}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}}$$

$$= \frac{\sin\left(\frac{B-C}{2}\right)}{\cos\frac{A}{2}}.$$

80. Let $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$, so $a = k\sin A$, $b = k\sin B$, and $c = k\sin C$.

Then $\frac{b+c}{a} = \frac{k(\sin B + \sin C)}{k\sin A} = \frac{\sin B + \sin C}{\sin A}$.

Using the sum-to-product formula, this becomes

$$\frac{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}{\sin A}. \quad \frac{(A+B+C)}{2} = 90° \Rightarrow$$

$$\frac{B+C}{2} = 90° - \frac{A}{2}, \text{so}$$

$$\sin\left(\frac{B+C}{2}\right) = \sin\left(90° - \frac{A}{2}\right) = \cos\frac{A}{2}.$$

Substituting and using the double angle formula in the denominator, we have

$$\frac{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}{\sin A} = \frac{2\cos\frac{A}{2}\cos\left(\frac{B-C}{2}\right)}{2\sin\frac{A}{2}\cos\frac{A}{2}} = \frac{\cos\left(\frac{B-C}{2}\right)}{\sin\frac{A}{2}}.$$

81. Let $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$, so $a = k\sin A$, $b = k\sin B$, and $c = k\sin C$.

$$\frac{b-c}{b+c} = \frac{k\sin B - k\sin C}{k\sin B + k\sin C} = \frac{\sin B - \sin C}{\sin B + \sin C}$$

$$= \frac{2\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)}{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}$$

$$= \cot\left(\frac{B+C}{2}\right)\tan\left(\frac{B-C}{2}\right)$$

$$= \frac{\tan\left(\frac{B-C}{2}\right)}{\tan\left(\frac{B+C}{2}\right)}$$

82. From exercise 81, we can show that

$$\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\frac{A}{2}.$$

$$\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\frac{A}{2}$$

$$= \frac{\left(\sqrt{3}+1\right)-\left(\sqrt{3}-1\right)}{\left(\sqrt{3}+1\right)+\left(\sqrt{3}-1\right)}\cot 30°$$

$$= \frac{2}{2\sqrt{3}}\cot 30° = 1$$

$$\tan\left(\frac{B-C}{2}\right) = 1 \Rightarrow \frac{B-C}{2} = \tan^{-1}1 = 45° \Rightarrow$$

$$B - C = 90°$$

7.1 Critical Thinking/Discussion/Writing

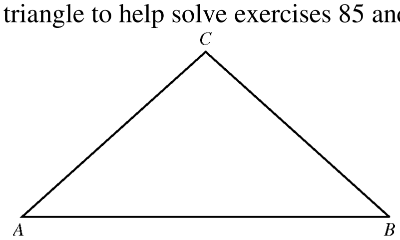
83. An isosceles triangle with $m\angle A = m\angle B \Rightarrow$

$a = b \Rightarrow \frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow a\sin B = b\sin A \Rightarrow$
$a\sin A = b\sin B$

84. An isosceles right triangle with right angle C gives

$m\angle A = m\angle B = 45° \Rightarrow$
$\sin A = \sin B = \cos A = \cos B \Rightarrow$
$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow a\sin B = b\sin A \Rightarrow$
$a\cos A = b\cos B$

Use the triangle to help solve exercises 85 and 86.



85. Given isosceles triangle ABC with $AB = BC$.

$\frac{\sin A}{BC} = \frac{\sin B}{AC} \Rightarrow \frac{\sin A}{AC} = \frac{\sin B}{AC} \Rightarrow$
$\sin A = \sin B \Rightarrow A = B$

86. Given triangle ABC with $m\angle A = m\angle B$.

$\frac{AC}{\sin\angle A} = \frac{BC}{\sin\angle B} \Rightarrow \frac{AC}{\sin\angle A} = \frac{BC}{\sin\angle A} \Rightarrow$
$AC = BC$

7.1 Maintaining Skills

87. $\cos 30° = \frac{\sqrt{3}}{2}$

88. $\cos 120° = -\frac{1}{2}$

89. $\cos 135° = -\frac{\sqrt{2}}{2}$

90. $\cos 150° = -\frac{\sqrt{3}}{2}$

91. $\cos^{-1}\left(\frac{1}{2}\right) = 60°$

92. $\cos^{-1}(2)$ is undefined because $-1 \le \cos\theta \le 1$.

93. $\cos^{-1}\left(-\frac{1}{2}\right) = 120°$

94. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = 150°$

95. $\sqrt{2^2 + 3^2 - 2(2)(3)\left(\frac{1}{6}\right)} = \sqrt{11}$

96. $\sqrt{4^2 + 5^2 - 2(4)(5)\left(\frac{1}{10}\right)} = \sqrt{37}$

97. $\sqrt{3^2 + 1^2 - 2(3)(1)\cos 60°} = \sqrt{10 - 6\left(\frac{1}{2}\right)} = \sqrt{7}$

98. $\sqrt{\left(\sqrt{8}\right)^2 + 4^2 - 2\left(\sqrt{8}\right)(4)\cos 45°}$

$$= \sqrt{24 - 8\left(\sqrt{8}\right)\left(\frac{\sqrt{2}}{2}\right)} = \sqrt{8} = 2\sqrt{2}$$

99. $\frac{2^2 + 3^2 - 4^2}{2(2)(3)} = -\frac{1}{4}$

100. $\frac{4^2 + 3^2 - 5^2}{2(4)(3)} = 0$

101. $\cos^{-1}\left(\frac{1^2 + \left(\sqrt{3}\right)^2 - 1^2}{2(1)\left(\sqrt{3}\right)}\right) = \cos^{-1}\left(\frac{3}{2\sqrt{3}}\right)$

$$= \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 30°$$

102. $\cos^{-1}\left(\frac{3^2 + 5^2 - 7^2}{2(3)(5)}\right) = \cos^{-1}\left(-\frac{15}{30}\right)$

$$= \cos^{-1}\left(-\frac{1}{2}\right) = 120°$$

7.2 The Law of Cosines

7.2 Practice Problems

1. Given: c = 25 in., a = 15 in., and B = 60º − an SAS case.
Step 1: Use the appropriate form of the Law of Cosines.

$$b = \sqrt{15^2 + 25^2 - 2(15)(25)\cos 60°} \approx 21.8$$

Step 2: Use the Law of Sines.

$$\frac{\sin A}{a} = \frac{\sin B}{b} \Rightarrow \frac{\sin A}{15} = \frac{\sin 60°}{21.8}$$

$$A = \sin^{-1}\left(\frac{15\sin 60°}{21.8}\right) \approx 36.6°$$

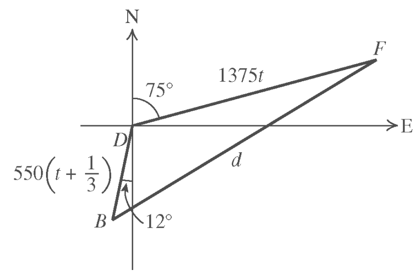
Step 3: Use the angle sum formula.

$$C = 180° - (60° + 36.6°) \approx 83.4°$$

Step 4: Write the solution.

$A \approx 36.6°$	a = 15 in.
$B = 60°$	**$b \approx 21.8$**
$C \approx 83.4°$	c = 25 in.

2.



The measure of $\angle FDB$ is $(90° - 75°) + 90° + 12° = 117°$. Then

$$d^2 = (1375t)^2 + \left[550\left(t + \frac{1}{3}\right)\right]^2 - 2(1375t)\left[550\left(t + \frac{1}{3}\right)\right]\cos 117°$$

Substitute $t = 3$ and then solve for d:

$$d^2 = (1375 \cdot 3)^2 + \left[550\left(3 + \frac{1}{3}\right)\right]^2 - 2(1375 \cdot 3)\left[550\left(3 + \frac{1}{3}\right)\right]\cos 117°$$

$$\approx 27,243,342.42$$

$$d \approx 5219.5 \text{ mi}$$

3. Given: $a = 4.5$, $b = 6.7$, and $c = 5.3$ − an SSS case
Step 1: Use the Law of Cosines to find the angle opposite the longest side.

$$6.7^2 = 4.5^2 + 5.3^2 - 2(4.5)(5.3)\cos B \Rightarrow$$

$$\cos B = \frac{4.5^2 + 5.3^2 - 6.7^2}{2(4.5)(5.3)} \approx 0.0723 \Rightarrow$$

$$B \approx 85.9°$$

Step 2: Use either the Law of Sines or the Law of Cosines again to find another angle.

$$\frac{\sin A}{4.5} = \frac{\sin 85.9°}{6.7} \Rightarrow$$

$$A \approx \sin^{-1}\left(\frac{4.5\sin 85.9°}{6.7}\right) \Rightarrow A \approx 42.1°$$

Step 3: Use the angle sum formula to find the third angle.

$$C \approx 180° - (85.9° + 42.1°) \approx 52.0°$$

(*continued on next page*)

(*continued*)

Step 4: Write the solution.

$\mathbf{A \approx 42.1°}$	$a = 4.5$
$\mathbf{B \approx 85.9°}$	$b = 6.7$
$\mathbf{C \approx 52.0°}$	$c = 5.3$

Note that answers will vary slightly if the computations are done in a different order.

4. Given: $a = 2$ in., $b = 3$ in., and $c = 6$ in. − an SSS case

$6^2 = 2^2 + 3^2 - 2(2)(3)\cos A \Rightarrow \cos A \approx -1.9167$

Since $0 \le \cos\theta \le 1$, no triangle exists.

7.2 Basic Concepts and Skills

1. One form of the Law of Cosines is $c^2 = a^2 + b^2 - 2ab\cos \underline{C}$.

2. If we take the angle in the Law of Cosines to be 90º, then we get the <u>Pythagorean</u> Theorem.

3. Triangles with SAS given are solve by the Law of Cosines, as are triangles with <u>three</u> sides given.

4. When one angle is found by the Law of Cosines, the other can be found with the Law of <u>Sines.</u>

5. False. Use the Law of Sines when two angles and a side are given.

6. True

7. $a = \sqrt{b^2 + c^2 - 2bc\cos A}$

$a = \sqrt{4^2 + 6^2 - 2(4)(6)\left(\frac{1}{16}\right)} = \sqrt{49} = 7$

8. $c = \sqrt{a^2 + b^2 - 2ab\cos C}$

$c = \sqrt{13^2 + 4^2 - 2(13)(4)\left(-\frac{5}{13}\right)} = \sqrt{225} = 15$

9. $c = \sqrt{a^2 + b^2 - 2ab\cos C}$

$c = \sqrt{5^2 + 3^2 - 2(5)(3)\cos 60°}$

$= \sqrt{34 - 30\left(\frac{1}{2}\right)} = \sqrt{34 - 15} = \sqrt{19}$

10. $b = \sqrt{a^2 + c^2 - 2ac\cos B}$

$b = \sqrt{6^2 + 8^2 - 2(6)(8)\cos 120°}$

$= \sqrt{100 - 96\left(-\frac{1}{2}\right)} = \sqrt{148} = 2\sqrt{37}$

11. $a^2 = b^2 + c^2 - 2bc\cos A \Rightarrow$

$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{6^2 + 5^2 - 7^2}{2(6)(5)} = \frac{1}{5}$

12. $c^2 = a^2 + b^2 - 2ab\cos C \Rightarrow$

$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{7^2 + 6^2 - 5^2}{2(7)(6)} = \frac{5}{7}$

13. $\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{4^2 + 5^2 - \left(\sqrt{21}\right)^2}{2(4)(5)}$

$= \frac{1}{2} \Rightarrow C = \cos^{-1}\left(\frac{1}{2}\right) = 60°$

14. $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$

$= \frac{7^2 + 5^2 - \left(\sqrt{109}\right)^2}{2(7)(5)} = -\frac{1}{2} \Rightarrow$

$B = \cos^{-1}\left(-\frac{1}{2}\right) = 120°$

Note that intermediate results are carried to one extra decimal place throughout the exercises. This leads to more accurate answers.

15. Given: $B = 106°, a = 14.6, c = 10.5$ − an SAS case.

$b = \sqrt{10.5^2 + 14.6^2 - 2(10.5)(14.6)\cos 106°}$
≈ 20.2

$A = \sin^{-1}\left(\frac{14.6\sin 106°}{20.2}\right) \approx 44°$

$C = 180° - (106° + 44°) \approx 30°$

16. Given: $A = 35°, b = 7.8, c = 5.6$ − an SAS case.

$a = \sqrt{7.8^2 + 5.6^2 - 2(7.8)(5.6)\cos 35°} \approx 4.5$

$C = \sin^{-1}\left(\frac{5.6\sin 35°}{\sqrt{7.8^2 + 5.6^2 - 2(7.8)(5.6)\cos 35°}}\right)$
$\approx 45°$

$B = 180° - (35° + 45°) \approx 100°$

17. Given: $a = 30, b = 18, c = 15$ − an SSS case.

$30^2 = 18^2 + 15^2 - 2(18)(15)\cos A \Rightarrow A \approx 130.5°$

$18^2 = 30^2 + 15^2 - 2(30)(15)\cos B \Rightarrow B \approx 27.1°$

$C = 180° - (130.5° + 27.1°) \approx 22.4°$

18. Given: $a = 9, b = 12, c = 7$ – an SSS case.
$9^2 = 12^2 + 7^2 - 2(12)(7)\cos A \Rightarrow A \approx 48.2°$
$12^2 = 9^2 + 7^2 - 2(12)(7)\cos B \Rightarrow B \approx 96.4°$
$C = 180° - (48.2° + 96.4°) \approx 35.4°$

19. Given: $a = 15, b = 9, C = 120°$ – an SAS case.
$c = \sqrt{15^2 + 9^2 - 2(15)(9)\cos 120°} = 21$
$\frac{\sin 120°}{21} = \frac{\sin A}{15} \Rightarrow A \approx 38.2°$
$B \approx 180° - (120° + 38.2°) \approx 21.8°$

20. Given: $a = 14, b = 10, C = 75°$ – an SAS case.
$c = \sqrt{14^2 + 10^2 - 2(14)(10)\cos 75°}$
$\approx 14.95 \approx 15.0$
$\frac{\sin 75°}{14.95} = \frac{\sin A}{14} \Rightarrow A \approx 64.8°$
$B \approx 180° - (64.8° + 75°) \approx 40.2°$

21. Given: $b = 10, c = 12, A = 62°$ – an SAS case.
$a = \sqrt{10^2 + 12^2 - 2(10)(12)\cos 62°} \approx 11.46 \approx 11.5$
$\frac{\sin 62°}{11.46} = \frac{\sin B}{10} \Rightarrow B \approx 50.4°$
$C \approx 180° - (62° + 50.2°) \approx 67.6°$

22. Given: $b = 11, c = 16, A = 110°$ – an SAS case.
$a = \sqrt{11^2 + 16^2 - 2(11)(16)\cos 110°}$
$\approx 22.30 \approx 22.3$
$\frac{\sin 110°}{22.30} = \frac{\sin B}{11} \Rightarrow B \approx 27.6°$
$C = 180° - (110° + 27.6°) \approx 42.4°$

23. Given: $c = 12, a = 15, b = 11$ – an SSS case
$15^2 = 11^2 + 12^2 - 2(11)(12)\cos A \Rightarrow A \approx 81.3°$
$\frac{\sin 81.3}{15} = \frac{\sin C}{12} \Rightarrow C \approx 52.3°$
$B \approx 180° - (81.3° + 52.3°) \approx 46.4°$

24. Given: $c = 16, a = 11, b = 13$ – an SSS case
$16^2 = 11^2 + 13^2 - 2(11)(13)\cos C \Rightarrow$
$C \approx 83.17° \approx 83.2°$
$\frac{\sin 83.17°}{16} = \frac{\sin B}{13} \Rightarrow B \approx 53.8°$
$A \approx 180° - (83.2° + 53.8°) \approx 43.0°$

25. Given: $a = 9, b = 13, c = 18$ – an SSS case
$18^2 = 9^2 + 13^2 - 2(9)(13)\cos C \Rightarrow$
$C \approx 108.44° \approx 108.4°$
$\frac{\sin 108.44°}{18} = \frac{\sin B}{13} \Rightarrow B \approx 43.3°$
$A \approx 180° - (43.3° + 108.4°) \approx 28.3°$

26. Given: $a = 14, b = 6, c = 10$ – an SSS case
$14^2 = 10^2 + 6^2 - 2(10)(6)\cos A \Rightarrow A = 120°$
$\frac{\sin 120°}{14} = \frac{\sin C}{10} \Rightarrow C \approx 38.2°$
$B \approx 180° - (120° + 38.2°) \approx 21.8°$

27. Given: $a = 2.5, b = 3.7, c = 5.4$ – an SSS case
$5.4^2 = 2.5^2 + 3.7^2 - 2(2.5)(3.7)\cos C \Rightarrow$
$C \approx 119.89° \approx 119.9°$
$\frac{\sin 119.89°}{5.4} = \frac{\sin B}{3.7} \Rightarrow B \approx 36.4°$
$A \approx 180° - (36.4° + 119.9°) \approx 23.7°$

28. Given: $a = 4.2, b = 2.9, c = 3.6$ – an SSS case
$4.2^2 = 2.9^2 + 3.6^2 - 2(2.9)(3.6)\cos A \Rightarrow$
$A \approx 79.71° \approx 79.7°$
$\frac{\sin 79.71°}{4.2} = \frac{\sin B}{2.9} \Rightarrow B \approx 42.8°$
$C \approx 180° - (79.7° + 42.8°) \approx 57.5°$

29. Given: $b = 3.2, c = 4.3, A = 97.7°$ – an SAS case
$a = \sqrt{3.2^2 + 4.3^2 - 2(3.2)(4.3)\cos 97.7°}$
$\approx 5.69 \approx 5.7$
$\frac{\sin 97.7°}{5.69} = \frac{\sin C}{4.3} \Rightarrow C \approx 48.5°$
$B \approx 180° - (97.7° + 48.5°) \approx 33.8°$

30. Given: $b = 5.4, c = 3.6, A = 79.2°$ – an SAS case
$a = \sqrt{5.4^2 + 3.6^2 - 2(5.4)(3.6)\cos 79.2°}$
$\approx 5.90 \approx 5.9$
$\frac{\sin 79.2°}{5.90} = \frac{\sin B}{5.4} \Rightarrow B \approx 64.0°$
$B \approx 180° - (79.2° + 64.0°) \approx 36.8°$

31. Given: $c = 4.9, a = 3.9, B = 68.3°$ – an SAS case $b = \sqrt{3.9^2 + 4.9^2 - 2(3.9)(4.9)\cos 68.3°}$
$\approx 5.01 \approx 5.0$
$\frac{\sin 68.3°}{5.01} = \frac{\sin A}{3.9} \Rightarrow A \approx 46.3°$
$C \approx 180° - (68.3° + 46.3°) \approx 65.4°$

32. Given: $c = 7.8, a = 9.8, B = 95.6°$ – an SAS case $b = \sqrt{7.8^2 + 9.8^2 - 2(7.8)(9.8)\cos 95.6°}$
$\approx 13.11 \approx 13.1$
$\frac{\sin 95.6°}{13.11} = \frac{\sin A}{9.8} \Rightarrow A \approx 48.1°$
$C \approx 180° - (95.6° + 48.1°) \approx 36.3°$

33. Given: $a = 2.3, b = 2.8, c = 3.7$ – an SSS case

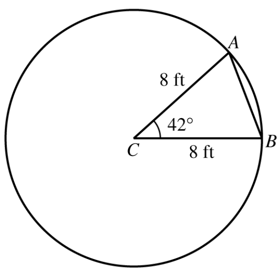
$3.7^2 = 2.3^2 + 2.8^2 - 2(2.3)(2.8)\cos C \Rightarrow$
$C \approx 92.49° \approx 92.5°$
$\frac{\sin 92.49°}{3.7} = \frac{\sin B}{2.8} \Rightarrow B \approx 49.1°$
$A \approx 180° - (92.5° + 49.1°) \approx 38.4°$

34. Given: $a = 5.3, b = 2.9, c = 4.6$ – an SSS case

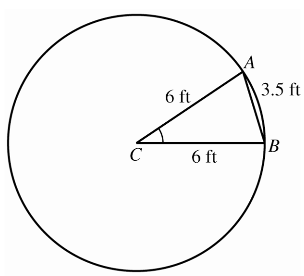
$5.3^2 = 2.9^2 + 4.6^2 - 2(2.9)(4.6)\cos A \Rightarrow$
$A \approx 86.82° \approx 86.8°$
$\frac{\sin 86.82°}{5.3} = \frac{\sin B}{2.9} \Rightarrow B \approx 33.1°$
$C \approx 180° - (86.8° + 33.1°) \approx 60.1°$

7.2 Applying the Concepts

35. $AB = \sqrt{8^2 + 8^2 - 2(8)(8)\cos 42°} \approx 5.7$ ft



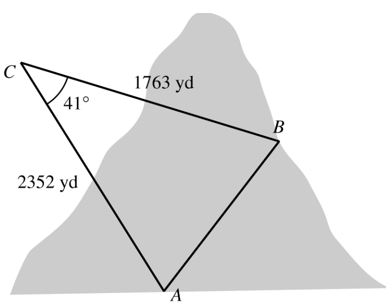
Exercise 35



Exercise 36

36. $3.5 = \sqrt{6^2 + 6^2 - 2(6)(6)\cos C} \Rightarrow C \approx 33.9°$

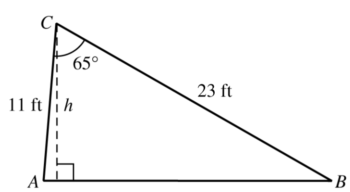
37.



$AB = \sqrt{2352^2 + 1763^2 - 2(2352)(1763)\cos 41°}$
≈ 1543.1 yd

38. $BC = \sqrt{537^2 + 823^2 - 2(537)(823)\cos 130°}$
≈ 1238.5 yd

39. a.



width $= AB = \sqrt{11^2 + 23^3 - 2(11)(23)\cos 65°}$
≈ 20.9 ft

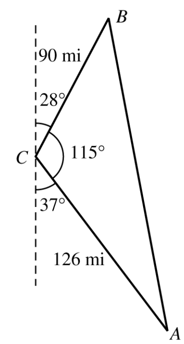
b. $\frac{\sin A}{23} = \frac{\sin 65°}{20.9} \Rightarrow \sin A = \frac{23\sin 65°}{20.9}$
$h = 11\sin A = 11\left(\frac{23\sin 65°}{20.9}\right) \approx 11.0$ ft

40. a. width $= AC = \sqrt{18^2 + 25^2 - 2(18)(25)\cos 55°}$
≈ 20.8 ft

b. $\frac{\sin \angle BAC}{25} = \frac{\sin 55°}{20.8} \Rightarrow \angle BAC \approx 79.9°$

41. $d = \sqrt{12.9^2 + 9^2 - 2(12.9)(9)\cos 45°} \approx 9.1$ mi

42.



Ship A traveled 7 hours at 18 mph, 126 miles. Ship B traveled 4.5 hours at 20 mph, 90 miles.

$m\angle ACB = 180° - (37° + 28°) = 115°$

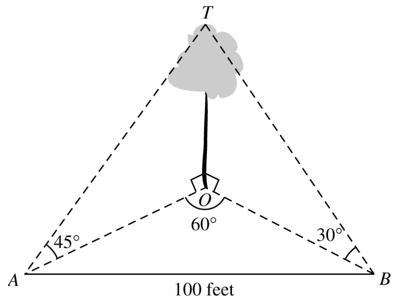
$AB = \sqrt{90^2 + 126^2 - 2(90)(126)\cos 115°}$
≈ 183.2 mi

43. a. $m\angle ABC = 180° - 63.7° = 116.3°$
$m\angle ACB = 21.4°$
$\frac{2000}{\sin 21.4°} = \frac{BC}{\sin 42.3°} \Rightarrow BC \approx 3689$ m

b. $\frac{2000}{\sin 21.4°} = \frac{AC}{\sin 116.3°} \Rightarrow AC \approx 4913.91$ m
$\cos 42.3° = \frac{2000 + DB}{4913.91} \Rightarrow DB \approx 1634.5$ m
The ship must travel about 1634.5 m farther.

44. $AB = 1.2 + 2.2 = 3.4$ in.
$BC = 2.2 + 3.1 = 5.3$ in.
$AC = 1.2 + 3.1 = 4.3$ in.
$AB = 3.4 = \sqrt{5.3^2 + 4.3^2 - 2(5.3)(4.3)\cos C} \Rightarrow$
$C \approx 39.8°$
$BC = 5.3 = \sqrt{3.4^2 + 4.3^2 - 2(4.3)(3.4)\cos A} \Rightarrow$
$A \approx 86.2°$
$B \approx 180° - (86.2° + 39.8°) \approx 54.0°$

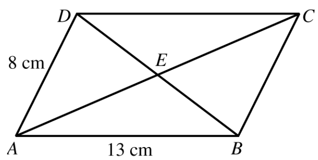
45.



In $\triangle AOT, AO = OT = x$. In $\triangle BOT$, $OB = OT\sqrt{3} = x\sqrt{3}$. So, in $\triangle BOA$, we have

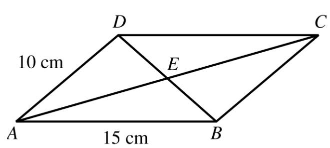
$$100^2 = x^2 + \left(x\sqrt{3}\right)^2 - 2x\left(x\sqrt{3}\right)\cos 60° \Rightarrow$$
$$100^2 = 4x^2 - \sqrt{3}x^2 = \left(4 - \sqrt{3}\right)x^2 \Rightarrow$$
$x \approx 66.4$ ft = the height of the tree

46.



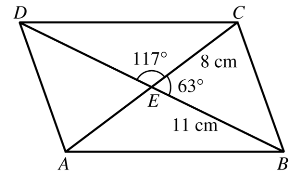
In $\triangle ABD$,
$$8^2 = 13^2 + 11^2 - 2(13)(11)\cos\angle ABD \Rightarrow$$
$m\angle ABD \approx 37.79°$.
In $\triangle BCD$,
$$13^2 = 8^2 + 11^2 - 2(8)(11)\cos\angle CBD \Rightarrow$$
$m\angle CBD \approx 84.78°$
$m\angle ABC = m\angle ABD + m\angle CBD \approx 122.57°$
In $\triangle ABC$,
$$AC = \sqrt{13^2 + 8^2 - 2(13)(8)\cos\angle ABC}$$
$$= \sqrt{13^2 + 8^2 - 2(13)(8)\cos 122.57°} \approx 18.6 \text{ cm}$$

47.



In $\triangle ABD, m\angle BAD = 40°$. So
$$BD = \sqrt{15^2 + 10^2 - 2(15)(10)\cos 40°} \approx 9.8 \text{ cm}$$
In $\triangle ABC, m\angle ABC = 180° - 40° = 140°$.
So,
$$AC = \sqrt{15^2 + 10^2 - 2(15)(10)\cos 140°}$$
$$\approx 23.6 \text{ cm}$$
The diagonals are 9.8 cm and 23.6 cm.

48.



The diagonals of a parallelogram bisect each other, so $BE = ED = 11$ and $AD = EC = 8$. In $\triangle BEC, BC = \sqrt{11^2 + 8^2 - 2(11)(8)\cos 63°}$
≈ 10.3 cm
In $\triangle CED, CD = \sqrt{11^2 + 8^2 - 2(11)(8)\cos 117°}$
≈ 16.3 cm
The diagonals are 10.3 cm and 16.3 cm.

7.2 Beyond the Basics

49. Using the Law of Cosines and solving for $\cos A$, we have

$$a^2 = b^2 + c^2 - 2bc\cos A \Rightarrow$$
$$\frac{a^2 - b^2 - c^2}{-2bc} = \cos A \Rightarrow$$
$$1 - \cos A = 1 - \left(-\frac{a^2 - b^2 - c^2}{2bc}\right)$$
$$= 1 + \left(\frac{a^2 - b^2 - c^2}{2bc}\right).$$

Expanding the right side, we have
$$\frac{(a-b+c)(a+b-c)}{2bc} = \frac{a^2 - b^2 - c^2 + 2bc}{2bc}$$
$$= 1 + \left(\frac{a^2 - b^2 - c^2}{2bc}\right).$$

Thus, $1 - \cos A = \dfrac{(a-b+c)(a+b-c)}{2bc}$.

50.
$$\frac{2(s-b)(s-c)}{bc}$$
$$= \frac{2\left(\frac{1}{2}(a+b+c) - b\right)\left(\frac{1}{2}(a+b+c) - c\right)}{bc}$$
$$= \frac{2\left(\frac{a}{2} - \frac{b}{2} + \frac{c}{2}\right)\left(\frac{a}{2} + \frac{b}{2} - \frac{c}{2}\right)}{bc}$$
$$= \frac{(a-b+c)(a+b-c)}{2bc}.$$

From exercise 49, we have
$$1 - \cos A = \frac{(a-b+c)(a+b-c)}{2bc} \Rightarrow$$
$$1 - \cos A = \frac{2(s-b)(s-c)}{bc}.$$

51. Using the Law of Cosines and solving for $\cos A$, we have $a^2 = b^2 + c^2 - 2bc\cos A \Rightarrow$

$$\frac{a^2 - b^2 - c^2}{-2bc} = \cos A \Rightarrow$$

$$1 + \cos A = 1 - \frac{a^2 - b^2 - c^2}{2bc}.$$

Expanding the right side of the identity gives

$$\frac{(b+c+a)(b+c-a)}{2bc} = \frac{-a^2 + b^2 + c^2 + 2bc}{2bc} = 1 - \left(\frac{a^2 - b^2 - c^2}{2bc}\right).$$

Thus, $1 + \cos A = \dfrac{(b+c+a)(b+c-a)}{2bc}.$

52.
$$\frac{2s(s-a)}{bc} = \frac{2\left(\frac{1}{2}(a+b+c)\right)\left(\frac{1}{2}(a+b+c) - a\right)}{bc} = \frac{(a+b+c)\left(\frac{b}{2} + \frac{c}{2} - \frac{a}{2}\right)}{bc} = \frac{(b+c+a)(b+c-a)}{2bc}.$$

From exercise 51, we have

$$1 + \cos A = \frac{(b+c-a)(b+c-a)}{2bc} \Rightarrow$$

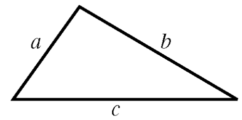
$$1 + \cos A = \frac{2s(s-a)}{bc}.$$

53.
$$\sin\frac{A}{2} = \sqrt{\frac{1 - \cos A}{2}} = \sqrt{\frac{\frac{2(s-b)(s-c)}{bc}}{2}} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

54.
$$\cos\frac{A}{2} = \sqrt{\frac{1 + \cos A}{2}} = \sqrt{\frac{\frac{2s(s-a)}{bc}}{2}} = \sqrt{\frac{s(s-a)}{bc}}$$

55.
$$\sin A = \sin 2\left(\frac{A}{2}\right) = 2\sin\frac{A}{2}\cos\frac{A}{2} = 2\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-a)}{bc}} = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

56.



The Law of Cosines gives

$$c^2 = b^2 + a^2 - 2ab\cos C \Rightarrow$$

$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$. Then

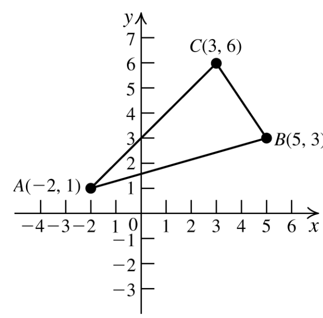
$$-1 < \cos C < 1$$
$$-1 < \frac{a^2 + b^2 - c^2}{2ab} < 1$$
$$-2ab < a^2 + b^2 - c^2 < 2ab$$
$$-a^2 - b^2 - 2ab < -c^2 < -a^2 - b^2 + 2ab$$
$$a^2 + 2ab + b^2 > c^2 > a^2 - 2ab + b^2$$
$$(a+b)^2 > c^2 > (a-b)^2$$

Since $a, b, c > 0$ and $a \le b$,

$$\sqrt{(a+b)^2} > \sqrt{c^2} > \sqrt{(b-a)^2}$$
$$a + b > c > b - a \Rightarrow$$
$$b - a < c < a + b$$

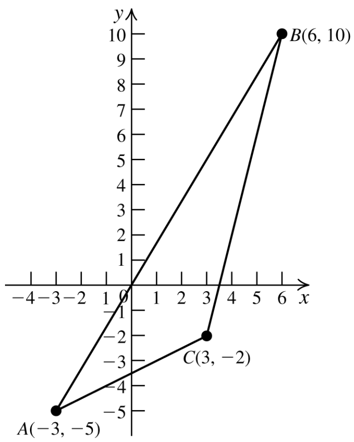
7.2 Critical Thinking/Discussion/Writing

57.



$$a = BC = \sqrt{(3-5)^2 + (6-3)^2} = \sqrt{13}$$
$$b = AC = \sqrt{(3-(-2))^2 + (6-1)^2} = 5\sqrt{2}$$
$$c = AB = \sqrt{(5-(-2))^2 + (3-1)^2} = \sqrt{53}$$
$$13 = \left(5\sqrt{2}\right)^2 + \left(\sqrt{53}\right)^2 - 2\left(5\sqrt{2}\right)\left(\sqrt{53}\right)\cos A \Rightarrow$$
$$A \approx 29.05° \approx 29.1°$$
$$\frac{\sin 29.05°}{\sqrt{13}} = \frac{\sin B}{5\sqrt{2}} \Rightarrow B \approx 72.2°$$
$$C \approx 180° - (29.1° + 72.2°) \approx 78.7°$$

58.



$a = BC = \sqrt{(6-3)^2 + (10-(-2))^2} = 3\sqrt{17}$
$b = AC = \sqrt{(3-(-3))^2 + (-2-(-5))^2} = 3\sqrt{5}$
$c = AB = \sqrt{(6-(-3))^2 + (10-(-5))^2} = 3\sqrt{34}$
$153 = \left(3\sqrt{5}\right)^2 + \left(3\sqrt{34}\right)^2 - 2\left(3\sqrt{5}\right)\left(3\sqrt{34}\right)\cos A \Rightarrow$
$A \approx 32.47° \approx 32.5°$
$\frac{\sin 32.47°}{3\sqrt{17}} = \frac{\sin B}{3\sqrt{5}} \Rightarrow B \approx 16.9°$
$C \approx 180° - (32.5° + 16.9°) \approx 130.6°$

59. Given: $B = 150°, b = 10, c = 6$
$b^2 = a^2 + c^2 - 2ac\cos B \Rightarrow$
$100 = a^2 + 36 - 12a\cos 150° \Rightarrow$
$100 = a^2 + 6\sqrt{3}a + 36 \Rightarrow$
$a^2 + 6\sqrt{3}a - 64 = 0 \Rightarrow$
$a = \frac{-6\sqrt{3} \pm \sqrt{\left(6\sqrt{3}\right)^2 - 4(1)(-64)}}{2} \approx 4.34 \approx 4.3$
(Note: the second solution is negative, so reject it. Only one triangle is possible.)
$4.34^2 = 10^2 + 6^2 - 2(10)(6)\cos A \Rightarrow$
$A \approx 12.48° \approx 12.5°$
$C \approx 180° - (150° - 12.5°) \approx 17.5°$

60. Given: $A = 30°, a = 6, b = 10$
$a^2 = b^2 + c^2 - 2bc\cos A \Rightarrow$
$36 = 100 + c^2 - 20c\cos 30° \Rightarrow$
$c^2 - 10\sqrt{3}c + 64 = 0 \Rightarrow$
$c = \frac{10\sqrt{3} \pm \sqrt{300 - 4(64)}}{2} \approx 11.98 \text{ or } c \approx 5.34$
So two triangles are possible.
If $c \approx 11.98, 11.98^2 = 36 + 100 - 120\cos C \Rightarrow$
$C \approx 93.6°; B \approx 180° - (30° + 93.6°) \approx 56.4°$

If $c \approx 5.34, 5.34^2 = 36 + 100 - 120\cos C \Rightarrow$
$C \approx 26.4°; B \approx 180° - (30° + 26.4°) \approx 123.6°$

61. Given: $A = 60°, a = 12, c = 15$
$a^2 = b^2 + c^2 - 2bc\cos A \Rightarrow$
$12^2 = b^2 + 15^2 - 30b\cos 60° \Rightarrow$
$b^2 - 15b + 81 = 0 \Rightarrow$
$b = \frac{-15 \pm \sqrt{225 - 4(81)}}{2} = \frac{-15 \pm \sqrt{-99}}{2}$
Both solutions are complex, so no triangle is possible.

62. $a^2 = b^2 + c^2 - 2bc\cos A \Rightarrow$
$b^2 + c^2 - 1 = b^2 + c^2 - 2bc\cos A \Rightarrow$
$1 = 2bc\cos A \Rightarrow \frac{1}{2} = bc\cos A$
Thus, if $b = c = 1$ and $A = 60°$, the above equation is true.

63. $a^2 = b^2 + c^2 - 2bc\cos A$ so that
$a^2 = b^2 + c^2 - 1$ is true for all values of $b = c$.

64. $a^2 < b^2 + c^2$ if angle A is an acute angle.

7.2 Maintaining Skills

For exercises 65−70, $a = 6$, $b = 4$, $c = 3$, $\alpha = 30°$, $\beta = 45°$, and $\theta = 60°$.

65. $b\sin\alpha = 4\sin 30° = 4\left(\frac{1}{2}\right) = 2$

66. $c\sin\beta = 3\sin 45° = \frac{3\sqrt{2}}{2}$

67. $\frac{1}{2}ab\sin\theta = \frac{1}{2}\cdot 6\cdot 4\sin 60° = 6\sqrt{3}$

68. $\frac{1}{2}bc\sin\alpha = \frac{1}{2}\cdot 4\cdot 3\sin 30° = 3$

69. $\frac{1}{2}ca\sin\beta = \frac{1}{2}\cdot 3\cdot 6\sin 45° = \frac{9\sqrt{2}}{2}$

70. $\frac{1}{2}ab\sin\alpha = \frac{1}{2}\cdot 6\cdot 4\sin 30° = 6$

71. a. $s = \frac{a+b+c}{2} = \frac{3+4+5}{2} = 6$

b. $K = \sqrt{s(s-a)(s-b)(s-c)}$
$= \sqrt{6(6-3)(6-4)(6-5)}$
$= \sqrt{6\cdot 3\cdot 2\cdot 1} = \sqrt{36} = 6$

72. a. $s = \dfrac{a+b+c}{2} = \dfrac{5+12+13}{2} = 15$

b. $K = \sqrt{s(s-a)(s-b)(s-c)}$
$= \sqrt{15(15-5)(15-12)(15-13)}$
$= \sqrt{15 \cdot 10 \cdot 3 \cdot 2} = \sqrt{900} = 30$

73. a. $s = \dfrac{a+b+c}{2} = \dfrac{6+6+6}{2} = 9$

b. $K = \sqrt{s(s-a)(s-b)(s-c)}$
$= \sqrt{9(9-6)(9-6)(9-6)}$
$= \sqrt{9 \cdot 3 \cdot 3 \cdot 3} = 9\sqrt{3}$

74. a. $s = \dfrac{a+b+c}{2} = \dfrac{15+11+6}{2} = 16$

b. $K = \sqrt{s(s-a)(s-b)(s-c)}$
$= \sqrt{16(16-15)(16-11)(16-6)}$
$= \sqrt{16 \cdot 1 \cdot 5 \cdot 10} = 20\sqrt{2}$

75. a. $s = \dfrac{a+b+c}{2} = \dfrac{18+10+14}{2} = 21$

b. $K = \sqrt{s(s-a)(s-b)(s-c)}$
$= \sqrt{21(21-18)(21-10)(21-14)}$
$= \sqrt{21 \cdot 3 \cdot 11 \cdot 7} = 21\sqrt{11}$

76. a. $s = \dfrac{a+b+c}{2} = \dfrac{12+17+25}{2} = 27$

b. $K = \sqrt{s(s-a)(s-b)(s-c)}$
$= \sqrt{27(27-12)(27-17)(27-25)}$
$= \sqrt{27 \cdot 15 \cdot 10 \cdot 2} = 90$

77. a. $s = \dfrac{a+b+c}{2} = \dfrac{10+13+13}{2} = 18$

b. $K = \sqrt{s(s-a)(s-b)(s-c)}$
$= \sqrt{18(18-10)(18-13)(18-13)}$
$= \sqrt{18 \cdot 8 \cdot 5 \cdot 5} = 60$

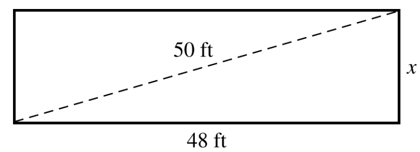
78. a. $s = \dfrac{a+b+c}{2} = \dfrac{26+28+30}{2} = 42$

b. $K = \sqrt{s(s-a)(s-b)(s-c)}$
$= \sqrt{42(42-26)(42-28)(42-30)}$
$= \sqrt{42 \cdot 16 \cdot 14 \cdot 12} = 336$

7.3 Areas of Polygons Using Trigonometry

7.3 Practice Problems

1.



$48^2 + x^2 = 50^2 \Rightarrow x^2 = 50^2 - 48^2 \Rightarrow$
$x = \sqrt{50^2 - 48^2} = \sqrt{2500 - 2304} = \sqrt{196} = 14$
The width of the rectangle is 14 ft.

a. The perimeter is 2(48 + 14) = 2(62) = 124 ft

b. The area is (48)(14) = 672 ft^2.

2. $K = \dfrac{1}{2} bc \sin\theta = \dfrac{1}{2}(27)(38)\sin 47°$
≈ 375.2 sq ft

3. $K = \dfrac{1}{2} ab \sin\theta$
$6 = \dfrac{1}{2} \cdot 4 \cdot 3 \sin\theta \Rightarrow 6 = 6\sin\theta \Rightarrow 1 = \sin\theta \Rightarrow$
$\theta = 90°$

4. First, find the third angle of the triangle.
$C = 180° - 63° - 74° = 43°$
We are given side *C*, so use the formula
$K = \dfrac{c^2 \sin A \sin B}{2 \sin C}$ to find the area.
$K = \dfrac{18^2 \sin 63° \sin 74°}{2 \sin 43°} \approx 203.4$ sq in.

5. Given: *a* = 11 m, *b* = 17 m, and *c* = 20 m
Then $s = \dfrac{11+17+20}{2} = 24$.
$K = \sqrt{24(24-11)(24-17)(24-20)}$
≈ 93.5 sq m

6. First find the surface area of the pool.
$s = \dfrac{25+30+33}{2} = 44$
$K = \sqrt{44(44-25)(44-30)(44-33)}$
≈ 358.8091 sq ft
The volume of the pool is
$358.8091 \cdot 5.5 \approx 1973.45$ cu ft.
One cubic foot contains approximately 7.5 gallons of water, so $1973.45 \times 7.5 \approx 14{,}801$ gal of water will fill the pool.

7.3 Basic Concepts and Skills

1. The area K of a triangle with base b and height h is $\underline{K = \frac{1}{2}bh}$.

2. An *SAS* triangle is one in which two sides and the included angle are known.

3. The area of an *SAS* triangle *ABC* with sides a and c is $\underline{K = \frac{1}{2}ac\sin B}$.

4. Heron's formula states that the area of an *SSS* triangle is $\underline{K = \sqrt{s(s-a)(s-b)(s-c)}}$, where $\underline{s = \frac{1}{2}(a+b+c)}$.

5. $K = \frac{1}{2}bc\sin A = \frac{1}{2}\cdot 6\cdot 5\sin 30° = \frac{1}{2}\cdot 6\cdot 5\cdot\frac{1}{2} = 7.5$

6. $K = \frac{1}{2}ac\sin B = \frac{1}{2}\cdot 12\cdot 7\sin 60° = \frac{1}{2}\cdot 12\cdot 7\cdot\frac{\sqrt{3}}{2} = 21\sqrt{3}$

7. $K = \frac{1}{2}ab\sin C = \frac{1}{2}\cdot 8\cdot 5\sin 120° = \frac{1}{2}\cdot 8\cdot 5\cdot\frac{\sqrt{3}}{2} = 10\sqrt{3}$

8. $K = \frac{1}{2}bc\sin A = \frac{1}{2}\cdot 4\cdot 6\sin 135° = \frac{1}{2}\cdot 4\cdot 6\cdot\frac{\sqrt{2}}{2} = 6\sqrt{2}$

9. $K = \frac{1}{2}ac\sin B = \frac{1}{2}\cdot 12\cdot 9\sin 150° = \frac{1}{2}\cdot 12\cdot 9\cdot\frac{1}{2} = 27$

10. $K = \frac{1}{2}ab\sin C = \frac{1}{2}\cdot\sqrt{8}\cdot 5\sin 45° = \frac{1}{2}\cdot 2\sqrt{2}\cdot 5\cdot\frac{\sqrt{2}}{2} = 5$

For exercises 11–18, use the formula $K = \frac{1}{2}bc\sin\theta$, where θ is the angle included between the sides of lengths b and c.

11. $K = \frac{1}{2}(30)(52)\sin 57° \approx 654.2 \text{ in.}^2$

12. $K = \frac{1}{2}(20)(27)\sin 110° \approx 253.7 \text{ cm}^2$

13. $K = \frac{1}{2}(15)(22)\sin 46° \approx 118.7 \text{ km}^2$

14. $A = \frac{1}{2}(16.7)(18)\sin 146.7° \approx 82.5 \text{ ft}^2$

15. $A = \frac{1}{2}(12)(16.7)\sin 38.6° \approx 62.5 \text{ mm}^2$

16. $A = \frac{1}{2}(151.6)(221.8)\sin 112.5° \approx 15{,}532.7 \text{ ft}^2$

17. $A = \frac{1}{2}(271)(194.3)\sin 107.3° \approx 25{,}136.6 \text{ ft}^2$

18. $A = \frac{1}{2}(15.7)(18.2)\sin 131.8° \approx 106.5 \text{ mm}^2$

In exercises 19–26, first find the measure of the third angle, then use one of the formulas listed on page 666 in the text.

19. $A = 180° - 57° - 49° = 74°$
We are given side a, so use the formula $K = \frac{a^2\sin B\sin C}{2\sin A}$ to find the area.
$K = \frac{16^2\sin 57°\sin 49°}{2\sin 74°} \approx 84.3$ sq ft

20. $B = 180° - 73° - 64° = 43°$
We are given side a, so use the formula $K = \frac{a^2\sin B\sin C}{2\sin A}$ to find the area.
$K = \frac{12^2\sin 43°\sin 64°}{2\sin 73°} \approx 46.2$ sq ft

21. $C = 180° - 64° - 38° = 78°$
We are given side b, so use the formula $K = \frac{b^2\sin C\sin A}{2\sin B}$ to find the area.
$K = \frac{15.3^2\sin 78°\sin 64°}{2\sin 38°} \approx 167.1$ sq yd

22. $A = 180° - 53.4° - 65.6° = 61°$
We are given side b, so use the formula $K = \frac{b^2\sin C\sin A}{2\sin B}$ to find the area.
$K = \frac{10^2\sin 65.6°\sin 61°}{2\sin 53.4°} \approx 49.6$ sq m

23. $A = 180° - 55° - 37.5° = 87.5°$
We are given side *c*, so use the formula
$K = \frac{c^2 \sin A \sin B}{2\sin C}$ to find the area.
$K = \frac{16.3^2 \sin 87.5° \sin 55°}{2\sin 37.5°} \approx 178.6$ sq cm

24. $C = 180° - 64° - 84.2° = 31.8°$
We are given side *c*, so use the formula
$K = \frac{c^2 \sin A \sin B}{2\sin C}$ to find the area.
$K = \frac{20.5^2 \sin 64° \sin 84.2°}{2\sin 31.8°} \approx 356.6$ sq ft

25. $C = 180° - 62°15' - 44°30' = 73°15'$
We are given side *a*, so use the formula
$K = \frac{a^2 \sin B \sin C}{2\sin A}$ to find the area.
$K = \frac{65.4^2 \sin 44°30' \sin 73°15'}{2\sin 62°15'} \approx 1621.9$ sq ft

26. $A = 180° - 56°18' - 37°36' = 86°6'$
We are given side *b*, so use the formula
$K = \frac{b^2 \sin C \sin A}{2\sin B}$ to find the area.
$K = \frac{24.3^2 \sin 37°36' \sin 86°6'}{2\sin 56°18'} \approx 216.0$ sq m

In exercises 27–34, use Heron's formula.

27. $s = (2+3+4)/2 = 4.5$
$K = \sqrt{4.5(4.5-2)(4.5-3)(4.5-4)} = 2.9$

28. $s = (50+100+130)/2 = 140$
$K = \sqrt{140(140-50)(140-100)(140-130)}$
$= 2245.0$

29. $s = (50+50+75)/2 = 87.5$
$K = \sqrt{87.5(87.5-50)(87.5-50)(87.5-75)}$
$= 1240.2$

30. $s = (100+100+125)/2 = 162.5$
$K = \sqrt{162.5(162.5-100)(162.5-100) \cdot (162.5-125)}$
$= 4878.9$

31. $s = (7.5+4.5+6.0)/2 = 9$
$K = \sqrt{9(9-7.5)(9-4.5)(9-6)} = 13.5$

32. $s = (8.5+5.1+4.5)/2 = 9.05$
$K = \sqrt{9.05(9.05-8.5)(9.05-5.1)(9.05-4.5)}$
≈ 9.5

33. $s = (3.7+5.1+4.2)/2 = 6.5$
$K = \sqrt{6.5(6.5-3.7)(6.5-5.1)(6.5-4.2)} = 7.7$

34. $s = (9.8+5.7+6.5)/2 = 11$
$K = \sqrt{11(11-9.8)(11-5.7)(11-6.5)} = 17.7$

For exercises 35−38, use the formula $K = \frac{1}{2}ab\sin\theta$, where θ is the angle included between the sides of lengths *a* and *b*.

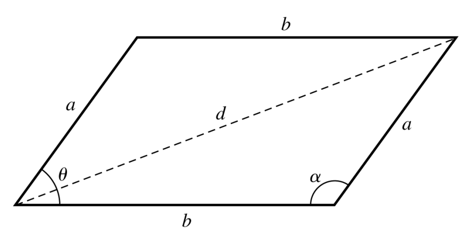
35. $12 = \frac{1}{2} \cdot 6 \cdot 8 \sin\theta \Rightarrow 12 = 24\sin\theta \Rightarrow$
$\sin\theta = \frac{1}{2} \Rightarrow \theta = \sin^{-1}\left(\frac{1}{2}\right) = 30°$ or
since $\sin\theta = \sin(180° - \theta)$,
$\theta = 180° - 30° = 150°$.

36. $6 = \frac{1}{2} \cdot 4 \cdot 2\sqrt{3}\sin\theta \Rightarrow 6 = 4\sqrt{3}\sin\theta \Rightarrow$
$\sin\theta = \frac{6}{4\sqrt{3}} = \frac{6\sqrt{3}}{12} = \frac{\sqrt{3}}{2} \Rightarrow$
$\theta = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = 60°$ or
since $\sin\theta = \sin(180° - \theta)$,
$\theta = 180° - 60° = 120°$.

37. $30 = \frac{1}{2} \cdot 12 \cdot 5\sin\theta \Rightarrow 30 = 30\sin\theta \Rightarrow$
$\sin\theta = 1 \Rightarrow \theta = 90°$

38. $15 = \frac{1}{2} \cdot 10\sqrt{2} \cdot 3\sin\theta \Rightarrow 15 = 15\sqrt{2}\sin\theta \Rightarrow$
$\sin\theta = \frac{15}{15\sqrt{2}} = \frac{\sqrt{2}}{2} \Rightarrow \theta = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = 45°$
or since $\sin\theta = \sin(180° - \theta)$,
$\theta = 180° - 45° = 135°$.

39.



Recall that opposite sides of a parallelogram have equal length. So the area of each triangle formed by the diagonal shown is $\frac{1}{2}ab\sin\alpha$, and the area of the parallelogram is
$2 \cdot \frac{1}{2}ab\sin\alpha = ab\sin\alpha$.

(*continued on next page*)

(*continued*)

From the diagram, we know that $90° < \alpha < 180° \Rightarrow \alpha$ lies in quadrant II, so $\sin\alpha = \sin(180° - \alpha) = \sin\theta$. Therefore, the area of the parallelogram can also be given by $ab\sin\theta$.

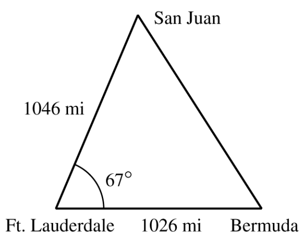
40. $K = ab\sin\theta = 8\cdot 13\sin 60° = 104\left(\frac{\sqrt{3}}{2}\right)$
$= 52\sqrt{3} \approx 90.1 \text{ cm}^2$

41. Since we are given the lengths of the sides and the diagonal, use Heron's formula to find the area of each of the triangles formed. The area of the parallelogram is twice the area of a triangle.
$s = \frac{12+16+22}{2} = 25$
$K_{\text{parallelogram}} = 2\sqrt{25(25-12)(25-16)(25-22)}$
$\approx 187.35 \text{ cm}^2$

42. $K = ab\sin\theta$
$10 = a\cdot 4\sin\frac{5\pi}{6} \Rightarrow 10 = a\cdot 4\cdot\frac{1}{2} \Rightarrow 10 = 2a \Rightarrow$
$a = 5$ cm

7.3 Applying the Concepts

43.



$A = \frac{1}{2}(1026)(1046)\sin 67° \approx 493{,}941 \text{ mi}^2$

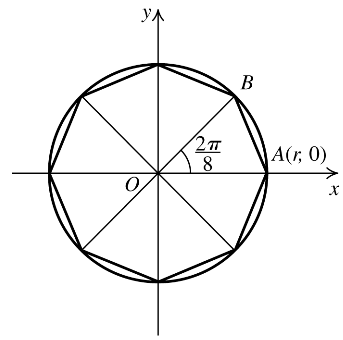
44. $A = \frac{1}{2}(20)(35)\sin 30° = 175 \text{ ft}^2$. At \$30 per square foot, the landscaping costs \$5250.

45. $s = (400+250+274)/2 = 462$
$K = \sqrt{462(462-400)(462-250)(462-274)}$
$\approx 33{,}788.1 \text{ ft}^2 \approx 0.775668$ acres
At \$1 million per acre, the lot is worth \$775,668.

46. $V = \text{height}\times\text{area of the base}$. Use Heron's formula to find the area of the base:
$s = (11+16+19)/2 = 23$
$K = \sqrt{23(23-11)(23-16)(23-19)} \approx 87.91 \text{ ft}^2$
$V = 87.91\times 5 = 439.55 \text{ ft}^3$.
There are $439.55\times 7.5 \approx 3297$ gal.

7.3 Beyond the Basics

Use this figure for exercises 47–50.

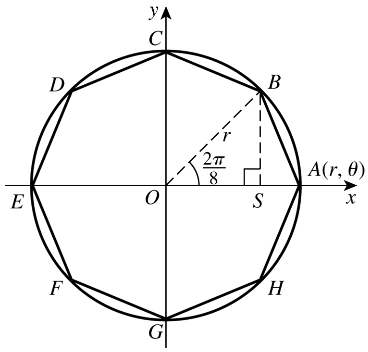


47. a. The coordinates of $A = (r, 0)$. Each central angle $= \frac{2\pi}{8} = \frac{\pi}{4}$. Thus, the coordinates of $B = \left(r\cos\frac{\pi}{4}, r\sin\frac{\pi}{4}\right) = \left(\frac{r\sqrt{2}}{2}, \frac{r\sqrt{2}}{2}\right)$.
Continuing in a counterclockwise direction, the coordinates of the vertices are $(0, r)$, $\left(-\frac{r\sqrt{2}}{2}, \frac{r\sqrt{2}}{2}\right)$, $(-r, 0)$, $\left(-\frac{r\sqrt{2}}{2}, -\frac{r\sqrt{2}}{2}\right)$, $(0, -r)$, $\left(\frac{r\sqrt{2}}{2}, -\frac{r\sqrt{2}}{2}\right)$.

b. Using the distance formula, we have
$$AB = \sqrt{\left(\frac{r\sqrt{2}}{2} - r\right)^2 + \left(\frac{r\sqrt{2}}{2} - 0\right)^2}$$
$$= \sqrt{\left(\frac{r^2}{2} - r^2\sqrt{2} + r^2\right) + \frac{r^2}{2}}$$
$$= \sqrt{2r^2 - r\sqrt{2}} = r\sqrt{2-\sqrt{2}}$$

c. The perimeter is $8AB = 8r\sqrt{2-\sqrt{2}}$.

48. a.



Drop a perpendicular, BS, from vertex B to side OA. Then we have

$$\sin\frac{2\pi}{8}=\sin\frac{\pi}{4}=\frac{BS}{r}\Rightarrow\frac{\sqrt{2}}{2}=\frac{BS}{r}\Rightarrow BS=\frac{r\sqrt{2}}{2}.$$

The area of triangle $AOB=\frac{1}{2}bh=\frac{1}{2}(OA)(BS)=\frac{1}{2}r\left(\frac{r\sqrt{2}}{2}\right)=\frac{r^2\sqrt{2}}{4}.$

b. The area of the octagon is 8 times the area of triangle AOB or $2r^2\sqrt{2}$.

49. a. In a regular n-gon, the coordinates of B can be given as $\left(r\cos\frac{2\pi}{n},\ r\sin\frac{2\pi}{n}\right)$. Then

$$AB=\sqrt{\left(r\cos\frac{2\pi}{n}-r\right)^2+\left(r\sin\frac{2\pi}{n}-0\right)^2}=\sqrt{\left(r^2\cos^2\frac{2\pi}{n}-2r^2\cos\frac{2\pi}{n}+r^2\right)+r^2\sin^2\frac{2\pi}{n}}$$
$$=\sqrt{r^2\cos^2\frac{2\pi}{n}+r^2\sin^2\frac{2\pi}{n}-2r^2\cos\frac{2\pi}{n}+r^2}=\sqrt{r^2\left(\cos^2\frac{2\pi}{n}+\sin^2\frac{2\pi}{n}\right)-2r^2\cos\frac{2\pi}{n}+r^2}$$
$$=\sqrt{2r^2-2r^2\cos\frac{2\pi}{n}}=r\sqrt{2-2\cos\frac{2\pi}{n}}$$

Thus, the perimeter $P=rn\sqrt{2-2\cos\frac{2\pi}{n}}$.

b. $C=2\pi r=10\pi\approx 31.4159$

c. $P=4\cdot 5\sqrt{2-2\cos\frac{2\pi}{4}}\approx 28.2843$

$P=10\cdot 5\sqrt{2-2\cos\frac{2\pi}{10}}\approx 30.9017$

$P=50\cdot 5\sqrt{2-2\cos\frac{2\pi}{50}}\approx 31.3953$

$P=100\cdot 5\sqrt{2-2\cos\frac{2\pi}{100}}\approx 31.4108$

The perimeters approach the circumference of the circle.

50. a. In a regular n-gon, $BS=\sin\frac{2\pi}{n}$. Then the area of triangle $AOB=\frac{1}{2}bh=\frac{1}{2}(OA)(BS)=\frac{r}{2}\left(r\sin\frac{2\pi}{n}\right)$. Therefore, the area of the n-gon $A=\frac{nr^2}{2}\sin\frac{2\pi}{n}$.

b. $A=\pi r^2=25\pi\approx 78.5398$

c. $A=\frac{25\cdot 4}{2}\sin\frac{2\pi}{4}\approx 50.0$

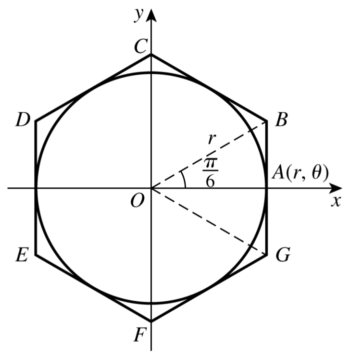
$A=\frac{25\cdot 10}{2}\sin\frac{2\pi}{10}\approx 73.4732$

$A=\frac{25\cdot 50}{2}\sin\frac{2\pi}{50}\approx 78.3333$

$A=\frac{25\cdot 100}{2}\sin\frac{2\pi}{100}\approx 78.4881$

The areas approach the area of the disc.

Use the figure to solve exercises 51−54.



51. a. From the figure, it is clear that the x-coordinate of B is r. Find the y-coordinate of B as follows:
$\tan\frac{\pi}{6} = \frac{AB}{r} \Rightarrow AB = r\tan\frac{\pi}{6}$. From geometry, we know that A is the midpoint of the side. Therefore, the length of each side of the hexagon is $2r\tan\frac{\pi}{6}$.

b. $P = 6\left(2r\tan\frac{\pi}{6}\right) = 12r\tan\frac{\pi}{6}$
$= 12r\left(\frac{\sqrt{3}}{3}\right) = 4r\sqrt{3}$

52. a. $A = \frac{1}{2}bh = \frac{1}{2}\left(r\tan\frac{\pi}{6}\right)r = \frac{1}{2}r^2\tan\frac{\pi}{6}$

b. The area of triangle $GOB = r^2\tan\frac{\pi}{6}$, so the area of the hexagon =
$6r^2\tan\frac{\pi}{6} = 6r^2\left(\frac{\sqrt{3}}{3}\right) = 2\sqrt{3}r^2$.

53. a. Using the same reasoning as in exercise 51, we know that the length of each side of the circumscribed n-gon is $2r\tan\frac{\pi}{n}$. Thus, the perimeter of the n-gon is $P = 2nr\tan\frac{\pi}{n}$.

b. $C = 2\pi r = 20\pi \approx 62.8319$

c. $P = 2(4)(10)\tan\frac{\pi}{4} \approx 80.0$
$P = 2(10)(10)\tan\frac{\pi}{10} \approx 64.9839$
$P = 2(50)(10)\tan\frac{\pi}{50} \approx 62.9147$
$P = 2(100)(10)\tan\frac{\pi}{100} \approx 62.8525$
The perimeters approach the perimeter of the circle.

54. a. Using the same reasoning as in exercise 48, we know that the area of the n-gon is
$nr^2\tan\frac{\pi}{n}$.

b. $A = \pi r^2 = 100\pi \approx 314.1593$

c. $A = (4)(10)^2\tan\frac{\pi}{4} = 400.0$
$A = (10)(10)^2\tan\frac{\pi}{10} \approx 324.9197$
$A = (50)(10)^2\tan\frac{\pi}{50} \approx 314.5733$
$A = (100)(10)^2\tan\frac{\pi}{100} \approx 314.2627$
The areas approach the area of the disc.

55. In triangle ABC,

$K = \frac{1}{2}ab\sin C$	Formula for area of a triangle
$4K^2 = a^2b^2\left(1-\cos^2 C\right)$	Multiply each side by 2 to obtain $2K = ab\sin C$. Square each side to obtain $(2K)^2 = (ab\sin C)^2 \Rightarrow 4K^2 = a^2b^2\sin^2 C.$ Replace $\sin^2 C$ with its equivalent $1-\cos^2\theta$ to obtain $4K^2 = a^2b^2\left(1-\cos^2 C\right)$
$4K^2 = a^2b^2\left[1-\left(\frac{a^2+b^2-c^2}{2ab}\right)^2\right]$	Start with the Law of Cosines and solve for cos C. $c^2 = a^2+b^2-2ab\cos C \Rightarrow$ $2ab\cos C = a^2+b^2-c^2 \Rightarrow$ $\cos C = \frac{a^2+b^2-c^2}{2ab} \Rightarrow$ $\cos^2\theta = \left(\frac{a^2+b^2-c^2}{2ab}\right)^2$
$16K^2 = 4a^2b^2-\left(a^2+b^2-c^2\right)^2$	On the right side, we have $a^2b^2\left[1-\left(\frac{a^2+b^2-c^2}{2ab}\right)^2\right]$ $= a^2b^2\left[1-\frac{\left(a^2+b^2-c^2\right)^2}{4a^2b^2}\right]$ $= a^2b^2-\frac{\left(a^2+b^2-c^2\right)^2}{4}$ Now multiply both sides of the equation by 4. $4K^2 = a^2b^2-\frac{\left(a^2+b^2-c^2\right)^2}{4} \Rightarrow$ $16K^2 = 4a^2b^2-\left(a^2+b^2-c^2\right)^2$
$16K^2 = \left[2ab+\left(a^2+b^2-c^2\right)\right]\left[2ab-\left(a^2+b^2-c^2\right)\right]$	Note that the right side of the equation is the difference of squares. Factoring gives $16K^2 = 4a^2b^2-\left(a^2+b^2-c^2\right)^2$ $= \left[2ab+\left(a^2+b^2-c^2\right)\right]\cdot$ $\left[2ab-\left(a^2+b^2-c^2\right)\right]$

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(*continued*)

$$16K^2 = \left[(a+b)^2 - c^2\right]\left[c^2 - (a-b)^2\right]$$

Rearrange terms inside the brackets on the right side.

$$\begin{aligned}16K^2 &= \left[a^2 + 2ab + b^2 - c^2\right]\cdot \\ &\qquad \left[-a^2 + 2ab - b^2 + c^2\right] \\ &= \left[(a+b)^2 - c^2\right]\left[-(a-b)^2 + c^2\right] \\ &= \left[(a+b)^2 - c^2\right]\left[c^2 - (a-b)^2\right]\end{aligned}$$

$$K^2 = \left(\frac{a+b+c}{2}\right)\left(\frac{a+b-c}{2}\right)\left(\frac{c+a-b}{2}\right)\left(\frac{c+b-a}{2}\right)$$

Again, we have the difference of squares on the right side.

$$\begin{aligned}16K^2 &= \left[(a+b)^2 - c^2\right]\left[c^2 - (a-b)^2\right] \\ &= \left[(a+b)+c\right]\left[(a+b)-c\right]\cdot \\ &\qquad \left[c+(a-b)\right]\left[c-(a-b)\right] \\ &= (a+b+c)(a+b-c)(c+a-b)(c+b-a)\end{aligned}$$

Now use the fact that $16 = 2^4$ to obtain

$$\begin{aligned}K^2 &= \frac{(a+b+c)(a+b-c)(c+a-b)(c+b-a)}{16} \\ &= \left(\frac{a+b+c}{2}\right)\left(\frac{a+b-c}{2}\right)\left(\frac{c+a-b}{2}\right)\left(\frac{c+b-a}{2}\right)\end{aligned}$$

$$K = \sqrt{s(s-a)(s-b)(s-c)}$$

Note that

$$s = \frac{a+b+c}{2},$$

$$s - a = \frac{a+b+c}{2} - a = \frac{a+b+c-2a}{2} = \frac{b+c-a}{2},$$

$$s - b = \frac{a+b+c}{2} - b = \frac{a+b+c-2b}{2} = \frac{a+c-b}{2},$$

$$s - c = \frac{a+b+c}{2} - c = \frac{a+b+c-2c}{2} = \frac{a+b-c}{2}.$$

Thus, we have

$$\begin{aligned}K^2 &= s\left(\frac{a+b-c}{2}\right)\left(\frac{c+a-b}{2}\right)\left(\frac{c+b-a}{2}\right) \\ &= s(s-c)(s-b)(s-a) \Rightarrow \\ K &= \sqrt{s(s-a)(s-b)(s-c)}\end{aligned}$$

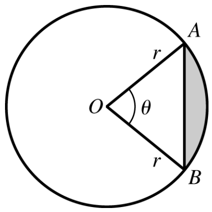
---

56. We know that $K = \frac{1}{2}bc\sin A$. Substitute Heron's formula for K and then solve for $\sin A$.

$$K = \sqrt{s(s-a)(s-b)(s-c)} = \frac{1}{2}bc\sin A \Rightarrow \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} = \sin A$$

57. From geometry, we know that the measure of each angle in an equilateral triangle is 60°. Thus, the area of an equilateral triangle whose sides have length a is $K = \frac{1}{2}a^2 \sin 60° = \frac{1}{2}a^2 \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}a^2$.

58. a.

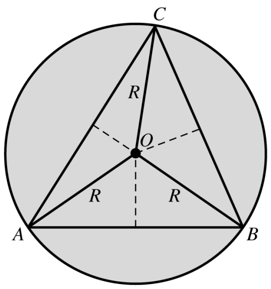


We are seeking the area of segment AOB. The area of sector AOB is given by $\pi r^2 \cdot \frac{\theta}{2\pi} = \frac{1}{2}r^2\theta.$ The area of triangle AOB is given by $K = \frac{1}{2}r^2 \sin\theta.$ Thus, the area of the segment is $K = \frac{1}{2}r^2\theta - \frac{1}{2}r^2 \sin\theta = \frac{1}{2}r^2(\theta - \sin\theta).$

b. $\theta = 60° = \frac{\pi}{3}$ radians

$$K = \frac{1}{2}r^2(\theta - \sin\theta) = \frac{1}{2}\cdot 6^2\left(\frac{\pi}{3} - \sin\frac{\pi}{3}\right) = 18\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) = \left(6\pi - 9\sqrt{3}\right) \text{ in.}^2$$

59.



We have $2A = \angle BOC,\ 2B = \angle AOC,$ and $2C = \angle AOB.$ We also have $a = BC$, $b = AC$, and $c = AB$. In $\triangle BOC,$ we have

$$a^2 = R^2 + R^2 - 2R\cdot R\cos\angle BOC \Rightarrow \cos\angle BOC = \frac{a^2 - R^2 - R^2}{-2R\cdot R} = \frac{R^2 + R^2 - a^2}{2R\cdot R} = \frac{2R^2 - a^2}{2R^2} = 1 - \frac{a^2}{2R^2} \Rightarrow$$

$$\cos 2A = 1 - \frac{a^2}{2R^2} \Rightarrow 1 - 2\sin^2 A = 1 - \frac{a^2}{2R^2} \Rightarrow \sin^2 A = \frac{a^2}{4R^2} \Rightarrow \sin A = \frac{a}{2R} \Rightarrow R = \frac{a}{2\sin A}$$

Similarly, we can show that $R = \frac{b}{2\sin B}$ and $R = \frac{c}{2\sin C}$. Therefore, $R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C}.$

60. Since the diameter is 1, the radius $R = \frac{1}{2}.$ From exercise 59, we have $R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} \Rightarrow$

$$\frac{1}{2} = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} \Rightarrow 1 = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow a = \sin A,\ b = \sin B, \text{ and } c = \sin C.$$

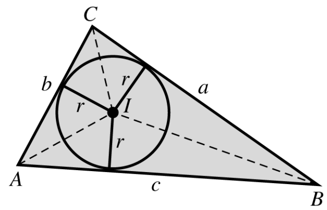
61. From exercise 56, we have $\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$

Using exercise 59, we have $R = \frac{a}{2\sin A} = \frac{a}{2\left(\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}\right)} = \frac{abc}{4K}.$

62. $s = \frac{1}{2}(18 + 24 + 30) = 36$

$K = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{36(36-18)(36-24)(36-30)} = 216$

Using exercise 61, we have $R = \frac{abc}{4K} = \frac{18\cdot 24\cdot 30}{4\cdot 216} = 15$

63. 

The area of $\triangle AIB$ is $\frac{1}{2}rc$. The area of $\triangle BIC$ is $\frac{1}{2}ra$. $\triangle AIC$ is $\frac{1}{2}rb$.

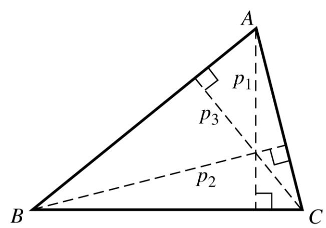
Thus,

$$K_{\triangle ABC} = \frac{1}{2}ra + \frac{1}{2}rb + \frac{1}{2}rc = \frac{1}{2}r(a+b+c) = rs \Rightarrow$$

$$\sqrt{s(s-a)(s-b)(s-c)} = rs \Rightarrow r = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}$$

64. Using the results from exercises 61 and 63, we have

$$rR = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} \cdot \frac{abc}{4K} = \frac{abc}{4s} = \frac{abc}{4\left[\frac{1}{2}(a+b+c)\right]} = \frac{abc}{2(a+b+c)}$$

65. 

$$\frac{1}{2}ap_1 = \frac{1}{2}bp_2 = \frac{1}{2}cp_3 = K \Rightarrow \frac{1}{p_1} = \frac{a}{2K},\quad \frac{1}{p_2} = \frac{b}{2K},\text{ and } \frac{1}{p_3} = \frac{c}{2K}.$$

Adding the three expressions, we have

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{a}{2K} + \frac{b}{2K} + \frac{c}{2K} = \frac{a+b+c}{2\sqrt{s(s-a)(s-b)(s-c)}} = \frac{2s}{2\sqrt{s(s-a)(s-b)(s-c)}}$$

$$= \frac{s}{\sqrt{s(s-a)(s-b)(s-c)}} \cdot \frac{\sqrt{s}}{\sqrt{s}} = \frac{s\sqrt{s}}{s\sqrt{(s-a)(s-b)(s-c)}} = \sqrt{\frac{s}{(s-a)(s-b)(s-c)}}$$

66. From exercises 61 and 64, we have

$$r = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} \Rightarrow$$

$$\frac{1}{r} = \frac{s}{\sqrt{s(s-a)(s-b)(s-c)}} \cdot \frac{\sqrt{s}}{\sqrt{s}} = \frac{s\sqrt{s}}{s\sqrt{(s-a)(s-b)(s-c)}} = \sqrt{\frac{s}{(s-a)(s-b)(s-c)}} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}$$

Thus, $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{r}$.

7.3 Critical Thinking/Discussion/Writing

67. Begin with $K = \frac{1}{2}ab\sin C.$

Since $C = 180° - (A + B)$, replace C with its equivalent.

$$K = \frac{1}{2}ab\sin\left[180-(A+B)\right] = \frac{1}{2}ab\sin(A+B) = \frac{ab\sin A\cos B + ab\sin B\cos A}{2}$$
$$= \frac{(b\sin A)a\cos B + b(a\sin B)\cos A}{2}$$
$$= \frac{2(a\sin B)a\cos B + 2b(b\sin A)\cos A}{4} \quad \text{(from the Law of Sines)}$$
$$= \frac{a^2\sin 2B + b^2\sin 2A}{4}$$

Similarly, we can prove $K = \frac{b^2\sin 2C + c^2\sin 2B}{4}$ and $K = \frac{c^2\sin 2A + b^2\sin 2C}{4}$.

68. From exercise 53 in section 7.2, we have

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}},\ \sin\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}},\ \text{and } \sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}.$$

Multiply these three equations to obtain

$$4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 4R\sqrt{\frac{(s-b)(s-c)}{bc}}\cdot\sqrt{\frac{(s-a)(s-c)}{ac}}\cdot\sqrt{\frac{(s-a)(s-b)}{ab}} = 4R\frac{(s-a)(s-b)(s-c)}{abc}$$
$$= 4\cdot\frac{abc}{4K}\cdot\frac{(s-a)(s-b)(s-c)}{abc} \quad \text{(from exercise 61)}$$
$$= \frac{(s-a)(s-b)(s-c)}{K} = \frac{(s-a)(s-b)(s-c)}{\sqrt{s(s-a)(s-b)(s-c)}}$$
$$= \frac{\left(\sqrt{(s-a)(s-b)(s-c)}\right)^2}{\sqrt{s(s-a)(s-b)(s-c)}} = \frac{\sqrt{(s-a)(s-b)(s-c)}}{\sqrt{s}} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} = r$$

(from exercise 63.)

7.3 Maintaining Skills

69. $2(-x+3y)+3(x-y) = -2x+6y+3x-3y$
$= x+3y \Rightarrow a = 1,\ b = 3$

$\sqrt{a^2+b^2} = \sqrt{1^2+3^2} = \sqrt{10}$

70. $3(2x-3y)-2(x-2y) = 6x-9y-2x+4y$
$= 4x-5y \Rightarrow$

$a = 4,\ b = -5$

$\sqrt{a^2+b^2} = \sqrt{4^2+(-5)^2} = \sqrt{41}$

71. a. $\sin\frac{\pi}{6} = \frac{1}{2}$

b. $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

c. $\tan\frac{\pi}{3} = \sqrt{3}$

72. a. $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

b. $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

c. $\tan^{-1}\left(-\sqrt{3}\right) = -\frac{\pi}{3}$

73. $\theta' = \tan^{-1}\left(\sqrt{3}\right) \Rightarrow \theta' = \frac{\pi}{3}$

In quadrant III, $\theta = \theta' + \pi = \frac{\pi}{3} + \pi = \frac{4\pi}{3}$

74. $\theta' = \left|\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right| \Rightarrow \theta' = \left|-\frac{\pi}{6}\right| = \frac{\pi}{6}$

In quadrant II, $\theta = \pi - \theta' = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

75. a. $d(S,T)=\sqrt{(6-1)^2+(15-3)^2}=\sqrt{25+144}$
$=13$
$d(O,P)=\sqrt{(5-0)^2+(12-0)^2}=\sqrt{25+144}$
$=13$

b. $m_1=\frac{15-3}{6-1}=\frac{12}{5}$
$m_2=\frac{12-0}{5-0}=\frac{12}{5}$

c. Since the slopes are equal, the lines are parallel.

76. a. $d(S,T)=\sqrt{(-1-2)^2+(-7-(-3))^2}$
$=\sqrt{(-3)^2+(-4)^2}=5$
$d(O,P)=\sqrt{(-3-0)^2+(4-0)^2}=5$

b. $m_1=\frac{-7-(-3)}{-1-2}=\frac{-4}{-3}=\frac{4}{3}$
$m_2=\frac{4-0}{-3-0}=-\frac{4}{3}$

c. Since the slopes are not equal, the lines are not parallel. Since the product of the slopes is not −1, the lines are not perpendicular.

77. a. $d(S,T)=\sqrt{(-2-(-1))^2+(1-2)^2}$
$=\sqrt{(-1)^2+(-1)^2}=\sqrt{2}$
$d(O,P)=\sqrt{(1-0)^2+(-1-0)^2}=\sqrt{2}$

b. $m_1=\frac{1-2}{-2-(-1)}=\frac{-1}{-1}=1$
$m_2=\frac{-1-0}{-0}=-1$

c. Since the product of the slopes is −1, the lines are perpendicular.

78. a. $d(S,T)=\sqrt{(3-(-1))^2+(-2-1)^2}$
$=\sqrt{4^2+(-3)^2}=5$
$d(O,P)=\sqrt{(4-0)^2+(-3-0)^2}=5$

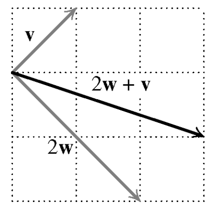
b. $m_1=\frac{-2-1}{3-(-1)}=-\frac{3}{4}$
$m_2=\frac{-3-0}{4-0}=-\frac{3}{4}$

c. Since the slopes are equal, the lines are parallel.

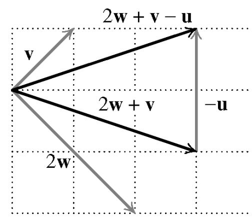
7.4 Vectors

7.4 Practice Problems

1. a.



b.



2. $\mathbf{w}=\langle 1-(-2),-3-7\rangle=\langle 3,-10\rangle$

3. $\mathbf{v}=\langle -1,2\rangle,\mathbf{w}=\langle 2,-3\rangle$

a. $\mathbf{v}+\mathbf{w}=\langle -1+2,2+(-3)\rangle=\langle 1,-1\rangle$

b. $3\mathbf{w}=3\langle 2,-3\rangle=\langle 3(2),3(-3)\rangle=\langle 6,-9\rangle$

c. $3\mathbf{w}-2\mathbf{v}=3\langle 2,-3\rangle-2\langle -1,2\rangle$
$=\langle 6,-9\rangle-\langle -2,4\rangle$
$=\langle 6-(-2),-9-4\rangle=\langle 8,-13\rangle$

d. $\|3\mathbf{w}-2\mathbf{v}\|=\|\langle 8,-13\rangle\|=\sqrt{8^2+(-13)^2}$
$=\sqrt{233}$

4. $\|\mathbf{v}\|=\|\langle -12,5\rangle\|=\sqrt{(-12)^2+5^2}=\sqrt{169}=13$
$\mathbf{u}=\frac{1}{\|\mathbf{v}\|}\mathbf{v}=\frac{1}{13}\langle -12,5\rangle=\left\langle -\frac{12}{13},\frac{5}{13}\right\rangle$

5. $\mathbf{u}=-3\mathbf{i}+2\mathbf{j},\ \mathbf{v}=\mathbf{i}+4\mathbf{j}$

a. $3\mathbf{u}+2\mathbf{v}=3(-3\mathbf{i}+2\mathbf{j})+2(\mathbf{i}+4\mathbf{j})$
$=-9\mathbf{i}+6\mathbf{j}+2\mathbf{i}+8\mathbf{j}=-7\mathbf{i}+14\mathbf{j}$

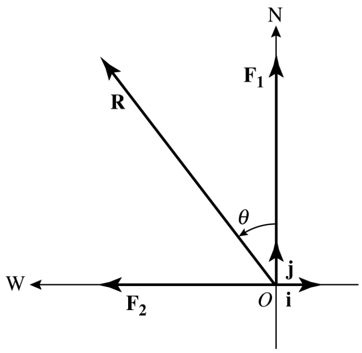
b. $\|3\mathbf{u}+2\mathbf{v}\|=\sqrt{(-7)^2+14^2}=\sqrt{245}=7\sqrt{5}$

6. $\mathbf{v}=2\left(\cos\frac{11\pi}{6}\mathbf{i}+\sin\frac{11\pi}{6}\mathbf{j}\right)=2\left(\frac{\sqrt{3}}{2}\mathbf{i}-\frac{1}{2}\mathbf{j}\right)$
$=\sqrt{3}\mathbf{i}-\mathbf{j}$

7. $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} = \langle 2, -3 \rangle \Rightarrow \tan\theta = \frac{-3}{2} = -\frac{3}{2}$

The reference angle θ' is given by $\theta' = \left|\tan^{-1}\left(-\frac{3}{2}\right)\right| \approx 56.31°$. Since the point (2, −3) lies in Quadrant IV, $\theta \approx 360° - 56.31° = 303.69°$.

8.



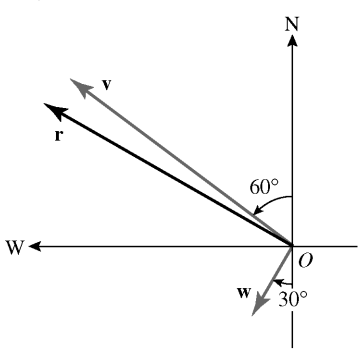
$\mathbf{F_1} = 40\mathbf{j}$ and $\mathbf{F}_2 = -30\mathbf{i}$. Then $\mathbf{R} = \mathbf{F_1} + \mathbf{F_2} = -30\mathbf{i} + 40\mathbf{j}$.

$\|\mathbf{R}\| = \sqrt{(-30)^2 + 40^2} = 50$

$\tan\theta' = -\frac{40}{30} \Rightarrow \theta' = \left|\tan^{-1}\left(-\frac{40}{30}\right)\right| \approx 53.13°$

The angle between **R** and the *y*-axis is $\theta \approx 90° - 53.1° = 36.9°$. Therefore, **R** is a force of 50 pounds in the direction N 36.9º W.

9. Let **v** be the velocity of the plane in still air. Let **w** be the wind speed, and let **r** be the resultant ground velocity of the plane. Since the bearing of the plane is N 60º W, the direction angle is 120º, and since the wind direction is S 30º W, its direction angle is 240º.



Not drawn to scale

$\mathbf{v} = 800(\cos 150°\mathbf{i} + \sin 150°\mathbf{j})$

$\mathbf{w} = 40(\cos 240°\mathbf{i} + \sin 240°\mathbf{j})$

$$\begin{aligned}\mathbf{r} &= \mathbf{v} + \mathbf{w}\\ &= 800(\cos 150°\mathbf{i} + \sin 150°\mathbf{j})\\ &\quad + 40(\cos 240°\mathbf{i} + \sin 240°\mathbf{j})\\ &= (800\cos 150° + 40\cos 240°)\mathbf{i}\\ &\quad + (800\sin 150° + 40\sin 240°)\mathbf{j}\end{aligned}$$

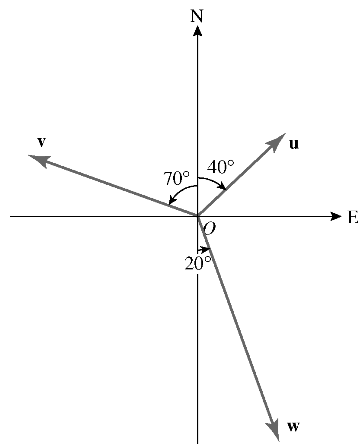
$$\begin{aligned}\|\mathbf{r}\| &= \sqrt{(800\cos 150° + 40\cos 240°)^2 + (800\sin 150° + 40\sin 240°)^2}\\ &= \sqrt{641{,}600} \approx 801.0\end{aligned}$$

Now find the direction angle of **r**.

$$\theta' = \left|\tan^{-1}\left(\frac{800\sin 150° + 40\sin 240°}{800\cos 150° + 40\cos 240°}\right)\right| \approx 27.1°$$

The angle between **r** and the *y*-axis is $\theta \approx 90° - 27.1° = 62.9°$. Therefore, the actual speed of the plane is approximately 801.0 miles per hour, and its bearing is approximately N 62.9º W.

10. **u** = 200 lb is the direction N 40º E, **v** = 300 lb in the direction N 70º W, and **w** = 400 lb in the direction S 20º E.



The horizontal component $\mathbf{R_1}$ of **R** is given by

$$\begin{aligned}\mathbf{R}_1 &= \mathbf{u}_1 + \mathbf{v}_1 + \mathbf{w}_1\\ &= 200\cos 50° + 300\cos 160° + 400\cos(-70°)\\ &\approx -16.5422\end{aligned}$$

The vertical component $\mathbf{R_1}$ of **R** is given by

$$\begin{aligned}\mathbf{R}_1 &= \mathbf{u}_1 + \mathbf{v}_1 + \mathbf{w}_1\\ &= 200\sin 50° + 300\sin 160° + 400\sin(-70°)\\ &\approx -120.06212\end{aligned}$$

$\mathbf{R} = \mathbf{R_1}\mathbf{i} + \mathbf{R_2}\mathbf{j} = -16.54\mathbf{i} - 120.06\mathbf{j}$

$\|\mathbf{R}\| = \sqrt{(-16.54)^2 + (-120.06)^2} \approx 121.2$

$\theta = \tan^{-1}\frac{-120.06}{-16.54} \approx 82.2°$

(*continued on next page*)

(*continued*)

From the diagram, we see that the resultant will lie in quadrant III. Thus, the magnitude of the resultant is about 121.2 lb at a bearing of about S 7.8º W.

7.4 Basic Concepts and Skills

1. A vector is a quantity that is characterized by a magnitude and a direction.

2. The resultant of **v** and **w** is the vector sum **v** + **w** and is represented by the diagonal of the parallelogram with adjacent sides **v** and **w**.

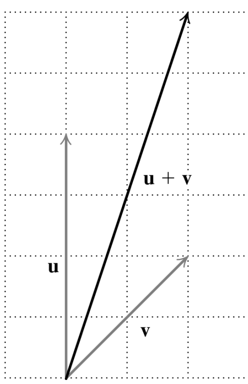
3. If $\mathbf{v} = \langle a, b \rangle$ and $a > 0$, then $\|\mathbf{v}\| = \sqrt{a^2 + b^2}$, and its direction angle $\theta = \tan^{-1}\left(\frac{b}{a}\right)$.

4. If **v** is a nonzero vector, then the unit vector **u** in the direction of **v** is given by $\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$.

5. True

6. True

7.



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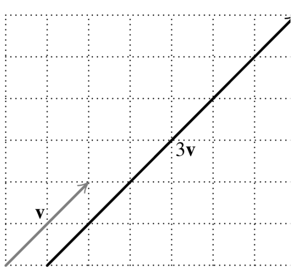
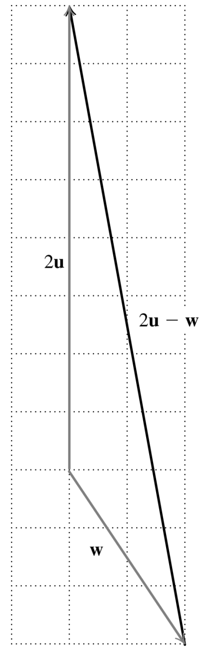
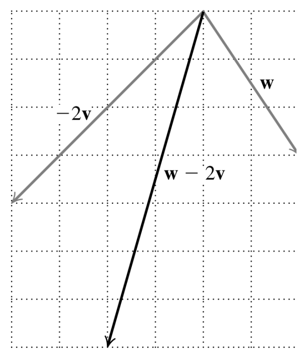


9.



10.



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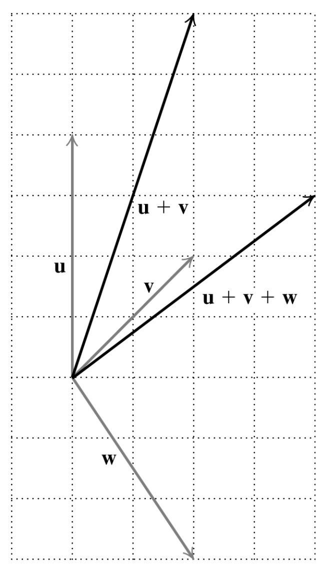
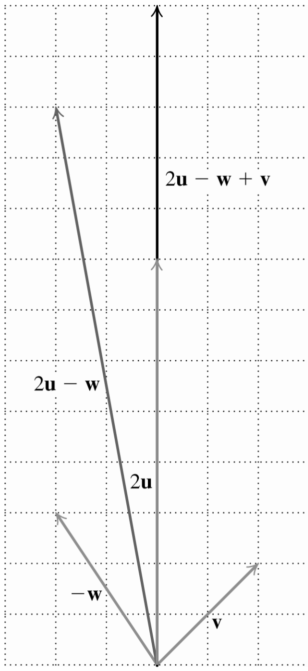
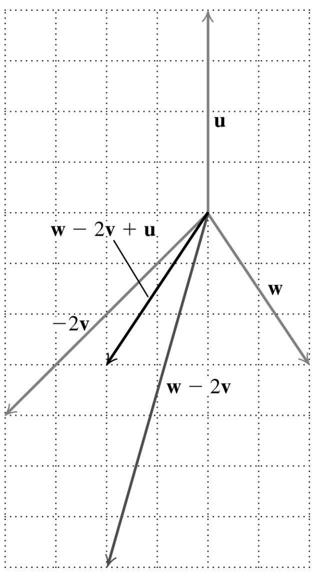


12.

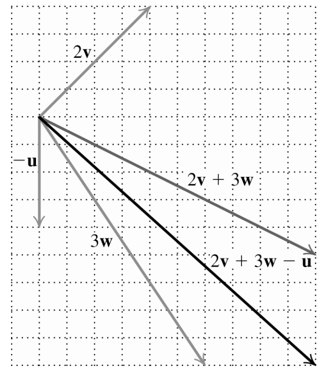


Note that the vectors $2\mathbf{u}$ and $\mathbf{2u - w + v}$ both start at the intersection of the vectors $-\mathbf{w}$ and $\mathbf{v}$.

13.



14.



15. $\mathbf{v} = \langle 2-3, 9-6 \rangle = \langle -1, 3 \rangle$

16. $\mathbf{v} = \langle 1-6, 1-(-4) \rangle = \langle -5, 5 \rangle$

17. $\mathbf{v} = \langle -3-(-5), -4-(-2) \rangle = \langle 2, -2 \rangle$

18. $\mathbf{v} = \langle -3-0, -6-0 \rangle = \langle -3, -6 \rangle$

19. $\mathbf{v} = \langle 2-(-1), -3-4 \rangle = \langle 3, -7 \rangle$

20. $\mathbf{v} = \langle -1.5-3.5, 1.3-2.7 \rangle = \langle -5, -1.4 \rangle$

21. $\mathbf{v} = \left\langle -\frac{1}{2}-\frac{1}{2}, -\frac{7}{4}-\frac{3}{4} \right\rangle = \left\langle -1, -\frac{5}{2} \right\rangle$

22. $\mathbf{v} = \left\langle \frac{1}{3}-\left(-\frac{2}{3}\right), -\frac{2}{3}-\frac{4}{9} \right\rangle = \left\langle 1, -\frac{10}{9} \right\rangle$

23. $\overrightarrow{AB} = \langle 3-1, 4-0 \rangle = \langle 2, 4 \rangle$
$\overrightarrow{CD} = \langle 1-(-1), 6-2 \rangle = \langle 2, 4 \rangle$
The vectors are equivalent.

24. $\overrightarrow{AB} = \langle 3-(-1), -2-2 \rangle = \langle 4, -4 \rangle$
$\overrightarrow{CD} = \langle 6-2, 1-5 \rangle = \langle 4, -4 \rangle$
The vectors are equivalent.

25. $\overrightarrow{AB} = \langle 3-2, 5-(-1) \rangle = \langle 1, 6 \rangle$
$\overrightarrow{CD} = \langle -2-(-1), -3-3 \rangle = \langle -1, -6 \rangle$
The vectors are not equivalent.

26. $\overrightarrow{AB} = \langle 6-5, 3-7 \rangle = \langle 1, -4 \rangle$
$\overrightarrow{CD} = \langle -3-(-2), 5-1 \rangle = \langle -1, 4 \rangle$
The vectors are not equivalent.

In exercises 27–34, $\mathbf{v} = \langle -1, 2 \rangle$ and $\mathbf{w} = \langle 3, -2 \rangle$.

27. $\|\mathbf{v}\| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$

28. $\|\mathbf{w}\| = \sqrt{3^2 + (-2)^2} = \sqrt{13}$

29. $\mathbf{v}-\mathbf{w}=\langle -1-3, 2-(-2)\rangle=\langle -4,4\rangle$

30. $\mathbf{v}+\mathbf{w}=\langle -1+3, 2+(-2)\rangle=\langle 2,0\rangle$

31. $2\mathbf{v}-3\mathbf{w}=2\langle -1,2\rangle-3\langle 3,-2\rangle$
$=\langle -2,4\rangle-\langle 9,-6\rangle=\langle -11,10\rangle$

32. $2\mathbf{w}-3\mathbf{v}=2\langle 3,-2\rangle-3\langle -1,2\rangle$
$=\langle 6,-4\rangle-\langle -3,6\rangle=\langle 9,-10\rangle$

33. $2\mathbf{v}-3\mathbf{w}=2\langle -1,2\rangle-3\langle 3,-2\rangle$
$=\langle -2,4\rangle-\langle 9,-6\rangle=\langle -11,10\rangle$
$\|2\mathbf{v}-3\mathbf{w}\|=\sqrt{(-11)^2+10^2}=\sqrt{221}$

34. $2\mathbf{w}-3\mathbf{v}=2\langle 3,-2\rangle-3\langle -1,2\rangle$
$=\langle 6,-4\rangle-\langle -3,6\rangle=\langle 9,-10\rangle$
$\|2\mathbf{w}-3\mathbf{v}\|=\sqrt{9^2+(-10)^2}=\sqrt{181}$

35. $\|\mathbf{v}\|=\sqrt{1^2+(-1)^2}=\sqrt{2}$
$\mathbf{u}=\frac{1}{\sqrt{2}}\langle 1,-1\rangle=\left\langle \frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right\rangle$

36. $\|\mathbf{v}\|=\sqrt{1^2+3^2}=\sqrt{10}$
$\mathbf{u}=\frac{1}{\sqrt{10}}\langle 1,3\rangle=\left\langle \frac{\sqrt{10}}{10},\frac{3\sqrt{10}}{10}\right\rangle$

37. $\|\mathbf{v}\|=\sqrt{(-4)^2+3^2}=5$
$\mathbf{u}=\frac{1}{5}\langle -4,3\rangle=\left\langle -\frac{4}{5},\frac{3}{5}\right\rangle$

38. $\|\mathbf{v}\|=\sqrt{5^2+(-12)^2}=13$
$\mathbf{u}=\frac{1}{13}\langle 5,-12\rangle=\left\langle \frac{5}{13},-\frac{12}{13}\right\rangle$

39. $\|\mathbf{v}\|=\sqrt{\left(\sqrt{2}\right)^2+\left(\sqrt{2}\right)^2}=2$
$\mathbf{u}=\frac{1}{2}\left\langle \sqrt{2},\sqrt{2}\right\rangle=\left\langle \frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right\rangle$

40. $\|\mathbf{v}\|=\sqrt{\left(\sqrt{5}\right)^2+(-2)^2}=3$
$\mathbf{u}=\frac{1}{3}\left\langle \sqrt{5},-2\right\rangle=\left\langle \frac{\sqrt{5}}{3},-\frac{2}{3}\right\rangle$

41. $\mathbf{u}+\mathbf{v}=(2\mathbf{i}-5\mathbf{j})+(-3\mathbf{i}-2\mathbf{j})=-\mathbf{i}-7\mathbf{j}$

42. $\mathbf{u}-\mathbf{v}=(2\mathbf{i}-5\mathbf{j})-(-3\mathbf{i}-2\mathbf{j})=5\mathbf{i}-3\mathbf{j}$

43. $2\mathbf{u}-3\mathbf{v}=2(2\mathbf{i}-5\mathbf{j})-3(-3\mathbf{i}-2\mathbf{j})=13\mathbf{i}-4\mathbf{j}$

44. $2\mathbf{v}+3\mathbf{u}=2(-3\mathbf{i}-2\mathbf{j})+3(2\mathbf{i}-5\mathbf{j})=-19\mathbf{j}$

45. $\|2\mathbf{u}-3\mathbf{v}\|=\|2(2\mathbf{i}-5\mathbf{j})-3(-3\mathbf{i}-2\mathbf{j})\|$
$=\|13\mathbf{i}-4\mathbf{j}\|=\sqrt{13^2+(-4)^2}=\sqrt{185}$

46. $\|2\mathbf{v}+3\mathbf{u}\|=\|2(-3\mathbf{i}-2\mathbf{j})+3(2\mathbf{i}-5\mathbf{j})\|=\|-19\mathbf{j}\|$
$=\sqrt{(-19)^2}=19$

47. $\mathbf{v}=2(\cos 30°\mathbf{i}+\sin 30°\mathbf{j})=2\left(\frac{\sqrt{3}}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}\right)$
$=\sqrt{3}\mathbf{i}+\mathbf{j}$

48. $\mathbf{v}=5(\cos 45°\mathbf{i}+\sin 45°\mathbf{j})=5\left(\frac{\sqrt{2}}{2}\mathbf{i}+\frac{\sqrt{2}}{2}\mathbf{j}\right)$
$=\frac{5\sqrt{2}}{2}\mathbf{i}+\frac{5\sqrt{2}}{2}\mathbf{j}$

49. $\mathbf{v}=4(\cos 120°\mathbf{i}+\sin 120°\mathbf{j})=4\left(-\frac{1}{2}\mathbf{i}+\frac{\sqrt{3}}{2}\mathbf{j}\right)$
$=-2\mathbf{i}+2\sqrt{3}\mathbf{j}$

50. $\mathbf{v}=3(\cos 150°\mathbf{i}+\sin 150°\mathbf{j})=3\left(-\frac{\sqrt{3}}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}\right)$
$=-\frac{3\sqrt{3}}{2}\mathbf{i}+\frac{3}{2}\mathbf{j}$

51. $\mathbf{v}=3\left(\cos\frac{5\pi}{3}\mathbf{i}+\sin\frac{5\pi}{3}\mathbf{j}\right)=3\left(\frac{1}{2}\mathbf{i}-\frac{\sqrt{3}}{2}\mathbf{j}\right)$
$=\frac{3}{2}\mathbf{i}-\frac{3\sqrt{3}}{2}\mathbf{j}$

52. $\mathbf{v}=4\left(\cos\frac{11\pi}{6}\mathbf{i}+\sin\frac{11\pi}{6}\mathbf{j}\right)=4\left(\frac{\sqrt{3}}{2}\mathbf{i}-\frac{1}{2}\mathbf{j}\right)$
$=2\sqrt{3}\mathbf{i}-2\mathbf{j}$

53. $\mathbf{v}=7\left(\cos\left(-\frac{\pi}{3}\right)\mathbf{i}+\sin\left(-\frac{\pi}{3}\right)\mathbf{j}\right)$
$=7\left(\frac{1}{2}\mathbf{i}-\frac{\sqrt{3}}{2}\mathbf{j}\right)=\frac{7}{2}\mathbf{i}-\frac{7\sqrt{3}}{2}\mathbf{j}$

54. $\mathbf{v}=8\left(\cos\frac{3\pi}{4}\mathbf{i}+\sin\frac{3\pi}{4}\mathbf{j}\right)=8\left(-\frac{\sqrt{2}}{2}\mathbf{i}+\frac{\sqrt{2}}{2}\mathbf{j}\right)$
$=-4\sqrt{2}\mathbf{i}+4\sqrt{2}\mathbf{j}$

55. $\|\mathbf{v}\|=10$; direction angle = 60º

56. $\|\mathbf{v}\|=4$; direction angle = 210º
Since the vector has a negative direction, the direction angle = 180º + 30º = 210º.

57. $\mathbf{v} = -3(\cos 30°\mathbf{i} - \sin 30°\mathbf{j}) = -3\left(\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}\right)$

$= -\frac{3\sqrt{3}}{2}\mathbf{i} + \frac{3}{2}\mathbf{j}$

Thus, the vector lies in quadrant II.

$\tan\theta = \frac{3/2}{-3\sqrt{3}/2} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3} \Rightarrow \theta = 150°$

$\|\mathbf{v}\| = 3$; direction angle = 150º

58. $\mathbf{v} = 2(\cos 300°\mathbf{i} - \sin 300°\mathbf{j}) = 2\left(\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}\right)$

$= \mathbf{i} + \sqrt{3}\mathbf{j}$

Thus, the vector lies in quadrant I.

$\tan\theta = \frac{\sqrt{3}}{1} \Rightarrow \theta = 60°$

$\|\mathbf{v}\| = 2$; direction angle = 60º

59. $\|\mathbf{v}\| = \sqrt{5^2 + 12^2} = 13$

$\theta = \tan^{-1}\frac{12}{5} \approx 67.38°$

60. $\|\mathbf{v}\| = \sqrt{12^2 + (-5)^2} = 13$

$\theta' = \left|\tan^{-1}\left(-\frac{5}{12}\right)\right| \approx 22.62°$

Since (12, −5) lies in Quadrant IV,
$\theta = 360° - 22.62° \approx 337.38°$.

61. $\|\mathbf{v}\| = \sqrt{(-4)^2 + (-3)^2} = 5$

$\theta' = \left|\tan^{-1}\left(\frac{-3}{-4}\right)\right| \approx 36.87°$

Since (−4, −3) lies in Quadrant III,
$\theta \approx 180° + 36.87° = 216.87°$.

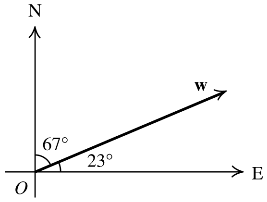
62. $\|\mathbf{v}\| = \sqrt{(-5)^2 + 12^2} = 13$

$\theta' = \left|\tan^{-1}\left(-\frac{12}{5}\right)\right| \approx 67.38°$

Since (−5, 12) lies in Quadrant II,
$\theta \approx 180° - 67.38° = 112.62°$.

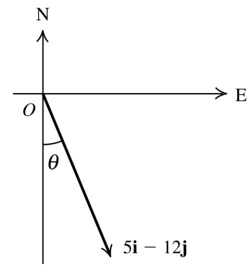
7.4 Applying the Concepts

63.



$\mathbf{w} = 25\cos 23°\mathbf{i} + 25\sin 23°\mathbf{j} \approx 23\mathbf{i} + 9.8\mathbf{j}$

64.



$\mathbf{v} = 5\mathbf{i} - 12\mathbf{j} \Rightarrow \|\mathbf{v}\| = 13$ mph

$5\mathbf{i} - 12\mathbf{j} = 13(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$
$= 13\cos\theta\mathbf{i} + 13\sin\theta\mathbf{j} \Rightarrow$

$5\mathbf{i} = 13\cos\theta\mathbf{i} \Rightarrow \theta = \cos^{-1}\left(\frac{5}{13}\right) \approx 67.4°$

$90° - 67.4° = 22.6°$, so the wind direction is S 22.6° E.

65. $\mathbf{F_1} = 0\mathbf{i} - 25\mathbf{j}$
$\mathbf{F_2} = -32\mathbf{i} + 0\mathbf{j}$
$\mathbf{R} = \mathbf{F_1} + \mathbf{F_2} = -32\mathbf{i} - 25\mathbf{j}$

$\|\mathbf{R}\| = \sqrt{(-32)^2 + (-25)^2} \approx 40.6$

$\theta' = \left|\tan^{-1}\left(\frac{-25}{-32}\right)\right| \approx 38.0°$

Since **R** lies in quadrant III,
$\theta \approx 180° + 38.0° = 218.0°$.
Thus, the resultant is a force of approximately 40.6 lb in the direction 218.0º.

66. $\mathbf{F_1} = 0\mathbf{i} - 50\mathbf{j}$
$\mathbf{F_2} = -32\mathbf{i} + 0\mathbf{j}$
$\mathbf{R} = \mathbf{F_1} + \mathbf{F_2} = -32\mathbf{i} - 50\mathbf{j}$

$\|\mathbf{R}\| = \sqrt{(-32)^2 + (-50)^2} \approx 59.4$

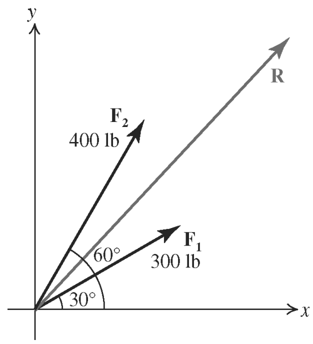
$\theta' = \left|\tan^{-1}\left(\frac{-50}{-32}\right)\right| \approx 57.38°$

Since **R** lies in quadrant III,
$\theta \approx 180° + 57.38° = 237.38°$.
Thus, the resultant is a force of approximately 59.4 lb in the direction 237.38º.

67. The direction angle of the force is 90º − 65º = 25º. Thus, the force
$\mathbf{v} = 80(\cos 25°\mathbf{i} + \sin 25°\mathbf{j})$
$\approx 72.50\mathbf{i} + 33.81\mathbf{j} \approx \langle 72.50, 33.81\rangle$.

68. The direction angle of the force is 270º − 32º = 238º. Thus, the force
$\mathbf{v} = 60(\cos 238°\ \mathbf{i} + \sin 283°\ \mathbf{j})$
$\approx -31.80\mathbf{i} - 50.88\mathbf{j} \approx \langle -31.80, -50.88\rangle$.

69.



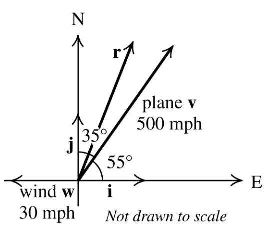
$\mathbf{F_1} = 300\cos 30°\mathbf{i} + 300\sin 30°\mathbf{j}$
$\mathbf{F_2} = 400\cos 60°\mathbf{i} + 400\sin 60°\mathbf{j}$
$\mathbf{R} = \mathbf{F_1} + \mathbf{F_2} \approx 459.81\mathbf{i} + 496.41\mathbf{j}$
$\|\mathbf{R}\| = \sqrt{(459.81)^2 + (496.41)^2} \approx 676.64$
$\theta' = \left|\tan^{-1}\left(\frac{496.41}{459.81}\right)\right| \approx 47.19°$

The resultant is a force of approximately 676.64 lb in the direction 47.19º.

70. $\mathbf{F_1} = 300\cos 60°\mathbf{i} + 300\sin 60°\mathbf{j}$
$\mathbf{F_2} = 400\cos 120°\mathbf{i} + 400\sin 120°\mathbf{j}$
$\mathbf{R} = \mathbf{F_1} + \mathbf{F_2} = -50\mathbf{i} + 606.22\mathbf{j}$
$\|\mathbf{R}\| = \sqrt{(-50)^2 + (606.22)^2} \approx 608.28$
$\theta' = \left|\tan^{-1}\left(\frac{606.22}{-50}\right)\right| \approx 85.285°$

Since **R** lies in quadrant II,
$\theta \approx 180° - 85.285° = 94.715° \approx 94.72°$.
Thus, the resultant is a force of approximately 608.28 lb in the direction 94.72º.

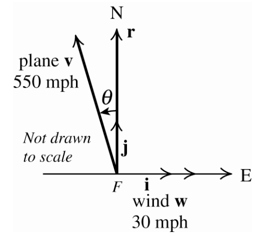
71.



$\mathbf{v} = 500\cos 55°\mathbf{i} + 500\sin 55°\mathbf{j}$
$\mathbf{w} = 30\cos 180°\mathbf{i} + 30\sin 180°\mathbf{j} = -30\mathbf{i}$
$\mathbf{r} = \mathbf{v} + \mathbf{w} = (500\cos 55° - 30)\mathbf{i} + 500\sin 55°\mathbf{j}$
$\approx 256.79\mathbf{i} + 409.58\mathbf{j}$
$\|\mathbf{r}\| = \sqrt{256.79^2 + 409.58^2} \approx 483.42$
≈ 483.4 mph
$\theta' = \left|\tan^{-1}\left(\frac{409.58}{256.79}\right)\right| \approx 57.9°$

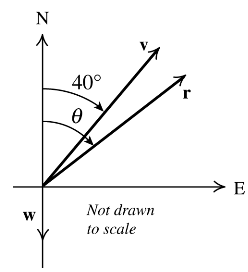
Since **R** lies in quadrant I,
$\theta \approx 90° - 57.9° = 32.1°$.
The plane's ground speed is 483.4 mph, and its bearing is $\text{N}\,32.1°\,\text{E}$.

72.



$\mathbf{v} = 550\cos(\theta + 90°)\mathbf{i} + 550\sin(\theta + 90°)\mathbf{j}$
$= -550\sin\theta\mathbf{i} + 550\cos\theta\mathbf{j}$
$\mathbf{w} = 30\cos 0°\mathbf{i} + 30\sin 0°\mathbf{j} = 30\mathbf{i}$
$\mathbf{r} = \mathbf{v} + \mathbf{w} = \langle 0,1\rangle$
$= (-550\sin\theta + 30)\mathbf{i} + 550\cos\theta\mathbf{j} \Rightarrow$
$-550\sin\theta\mathbf{i} + 30\mathbf{i} = 0 \Rightarrow \sin\theta = \frac{30}{550} \Rightarrow$
$\theta = \sin^{-1}\left(\frac{30}{550}\right) \approx 3.1° \Rightarrow$
The plane's bearing is N 3.1°W.

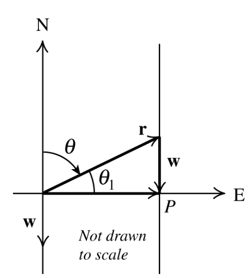
73.



$\mathbf{v} = 15\cos 50°\mathbf{i} + 15\sin 50°\mathbf{j}$
$\mathbf{w} = -4\mathbf{j}$
$\mathbf{r} = \mathbf{v} + \mathbf{w} = 15\cos 50°\mathbf{i} + (15\sin 50° - 4)\mathbf{j}$
$\|\mathbf{r}\| = \sqrt{(15\cos 50°)^2 + (15\sin 50° - 4)^2}$
$\approx 12.21 \approx 12.2$ mph
$\theta' = \left|\tan^{-1}\left(\frac{15\sin 50° - 4}{15\cos 50°}\right)\right| \approx 37.8°$

Since **R** lies in quadrant I,
$\theta \approx 90° - 37.8° = 52.2°$.
The boat's speed and direction are 12.2 mph and $\text{N}\,52.2°\,\text{E}$.

74.



$\sin\theta_1 = \frac{4}{15} \Rightarrow \theta_1 = \sin^{-1}\left(\frac{4}{15}\right) \approx 15.5° \Rightarrow$
$\theta = \text{N}\,74.5°\,\text{E}$

75. First find the resultant **R** of the system.

$\mathbf{u} = 30(\cos 26°\mathbf{i} + \sin 26°\mathbf{j})$
$\mathbf{v} = 40(\cos 115°\mathbf{i} + \sin 115°\mathbf{j})$
$\mathbf{w} = 50(\cos 270°\mathbf{i} + \sin 270°\mathbf{j})$

$\mathbf{R}_1 = 30\cos 26° + 40\cos 115° + 50\cos 270° \approx 10.06$

$\mathbf{R}_2 = 30\sin 26° + 40\sin 115° + 50\sin 270° \approx -0.60$

$\|\mathbf{R}\| = \sqrt{10.06^2 + (-0.60)^2} \approx 10.08$

$\theta' = \left|\tan^{-1}\frac{\mathbf{R}_2}{\mathbf{R}_1}\right| \approx 3.39°$

Since **R** lies in Quadrant IV, $\theta \approx 360° - 3.39° = 356.61°.$ The force that must be added to the system to obtain equilibrium is **–R**, a force of 10.08 pounds at an angle of 180º – 3.39º = 176.61º with the positive *x*-axis.

76. First find the resultant **R** of the system.

$\mathbf{u} = 15(\cos 210°\mathbf{i} + \sin 210°\mathbf{j})$
$\mathbf{v} = 40(\cos 90°\mathbf{i} + \sin 90°\mathbf{j})$
$\mathbf{w} = 25(\cos 310°\mathbf{i} + \sin 310°\mathbf{j})$

$\mathbf{R}_1 = 15\cos 210° + 40\cos 90° + 25\cos 310° \approx 3.08$

$\mathbf{R}_2 = 15\sin 210° + 40\sin 90° + 25\sin 310° \approx 13.35$

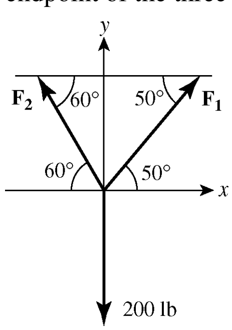
$\|\mathbf{R}\| = \sqrt{3.08^2 + 13.35^2} \approx 13.70$

$\theta' = \left|\tan^{-1}\left(\frac{13.35}{3.08}\right)\right| \approx 77.01°$

Since **R** lies in Quadrant I, $\theta \approx 77.01°.$ The force that must be added to the system to obtain equilibrium is **–R**, a force of 13.70 pounds at an angle of 180º + 77.07º = 257.01º with the positive *x*-axis.

7.4 Beyond the Basics

77. To help visualize the system, overlay a coordinate system with the origin at the endpoint of the three vectors.



Since the system is in equilibrium, the resultant **R** is 200(cos 270º**i** + sin 270º**j**) = 0**i** –200**j**.

$\mathbf{u} = \|\mathbf{F_1}\|(\cos 50°\mathbf{i} + \sin 50°\mathbf{j})$
$= \|\mathbf{F_1}\|\cos 50°\mathbf{i} + \|\mathbf{F_1}\|\sin 50°\mathbf{j}$
$\mathbf{v} = \|\mathbf{F_2}\|(\cos 120°\mathbf{i} + \sin 120°\mathbf{j})$
$= \|\mathbf{F_2}\|\cos 120°\mathbf{i} + \|\mathbf{F_2}\|\sin 120°\mathbf{j}$

Now solve the system

$\|\mathbf{F_1}\|\cos 50° + \|\mathbf{F_2}\|\cos 120° = 0$
$\|\mathbf{F_1}\|\sin 50° + \|\mathbf{F_2}\|\sin 120° = -200$

We will solve by elimination.

$\|\mathbf{F_1}\|\cos 50° - \frac{1}{2}\|\mathbf{F_2}\| = 0$

$\|\mathbf{F_1}\|\sin 50° + \frac{\sqrt{3}}{2}\|\mathbf{F_2}\| = -200$

$\|\mathbf{F_1}\|\sqrt{3}\cos 50° - \frac{\sqrt{3}}{2}\|\mathbf{F_2}\| = 0$

$\|\mathbf{F_1}\|\sin 50° + \frac{\sqrt{3}}{2}\|\mathbf{F_2}\| = -200$

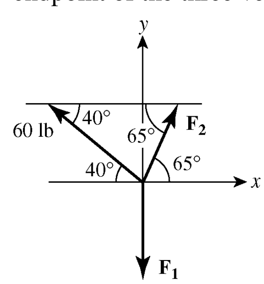
$\|\mathbf{F_1}\|\sqrt{3}\cos 50° + x\sin 50° = 200$

$\|\mathbf{F_1}\|\left(\sqrt{3}\cos 50° + \sin 50°\right) = 200 \Rightarrow$

$\|\mathbf{F_1}\| = \left|\frac{200}{\sqrt{3}\cos 50° + \sin 50°}\right| \approx 106.42 \text{ lb}$

$106.42\cos 50° + \|\mathbf{F_2}\|\cos 120° = 0 \Rightarrow$
$\|\mathbf{F_2}\| \approx 136.81 \text{ lb}$

78. To help visualize the system, overlay a coordinate system with the origin at the endpoint of the three vectors.



Since the system is in equilibrium, the resultant **R** is $\|\mathbf{F_1}\|(\cos 270°\mathbf{i} + \sin 270°\mathbf{j}) = 0\mathbf{i} - \|\mathbf{F_1}\|\mathbf{j}.$

$\mathbf{u} = \|\mathbf{F_2}\|(\cos 65°\mathbf{i} + \sin 65°\mathbf{j})$
$= \|\mathbf{F_2}\|\cos 65°\mathbf{i} + \|\mathbf{F_2}\|\sin 65°\mathbf{j}$
$\mathbf{v} = 60(\cos 140°\mathbf{i} + \sin 140°\mathbf{j})$
$= 60\cos 140°\mathbf{i} + 60\sin 120°\mathbf{j}$

(*continued on next page*)

(*continued*)

Now solve the system

$$\|\mathbf{F_2}\|\cos 65° + 60\cos 140° = 0$$
$$\|\mathbf{F_2}\|\sin 65° + 60\sin 140° = -\|\mathbf{F_1}\|$$

Solve the first equation for $\|\mathbf{F_2}\|$.

$$\|\mathbf{F_2}\|\cos 65° + 60\cos 140° = 0$$
$$\|\mathbf{F_2}\| = \frac{-60\cos 140°}{\cos 65°} \approx 108.76 \text{ lb}$$

Now substitute the value of $\|\mathbf{F_2}\|$ into the second equation and solve for $\|\mathbf{F_1}\|$.

$$108.76\sin 65° + 60\sin 140° = -\|\mathbf{F_1}\|$$
$$\|\mathbf{F_1}\| \approx 137.14 \text{ lb}$$

79. $\mathbf{v} = \langle 4,-3\rangle; P = (-2,1)$
$Q_x - (-2) = 4 \Rightarrow Q_x = 2$
$Q_y - 1 = -3 \Rightarrow Q_y = -2$
The terminal point is $(2,-2)$.

80. $\mathbf{v} = \langle -3,5\rangle; Q = (4,0)$
$4 - P_x = -3 \Rightarrow P_x = 7$
$0 - P_y = 5 \Rightarrow P_y = -5$
The initial point is $(7,-5)$.

81. $\mathbf{u} = \langle -1,2\rangle, \mathbf{v} = \langle 3,5\rangle$
$2\mathbf{u} - \mathbf{x} = 2\mathbf{x} + 3\mathbf{v} \Rightarrow 3\mathbf{x} = 2\mathbf{u} - 3\mathbf{v} \Rightarrow$
$3\mathbf{x} = 2\langle -1,2\rangle - 3\langle 3,5\rangle = \langle -11,-11\rangle \Rightarrow$
$\mathbf{x} = \left\langle -\frac{11}{3}, -\frac{11}{3}\right\rangle$

82. Let $R(x, y)$ be the coordinates of the point.
Then $\overrightarrow{PR} = \frac{3}{4}\overrightarrow{PQ}$. $\overrightarrow{PR} = \langle x-3, y-5\rangle$ and
$\overrightarrow{PQ} = \langle 7-3, -4-5\rangle = \langle 4,-9\rangle$.
$\langle x-3, y-5\rangle = \frac{3}{4}\langle 4,-9\rangle = \left\langle 3, -\frac{27}{4}\right\rangle \Rightarrow$
$x - 3 = 3 \Rightarrow x = 6$
$y - 5 = -\frac{27}{4} \Rightarrow y = -\frac{7}{4}$
The coordinates of R are $\left(6, -\frac{7}{4}\right)$.

83. $\overrightarrow{AB} = \langle 1-2,\ 4-3\rangle = \langle -1,\ 1\rangle = -\mathbf{i} + \mathbf{j}$
$\overrightarrow{CD} = \langle 1-0,\ -3-(-2)\rangle = \langle 1,\ -1\rangle = \mathbf{i} - \mathbf{j}$
Thus, $\overrightarrow{AB} = -\overrightarrow{CD}$, so $\overrightarrow{AB} \parallel -\overrightarrow{CD}$.
$\|\overrightarrow{AB}\| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$
$\|\overrightarrow{CD}\| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$
From geometry, we know that if a quadrilateral has two opposite sides that are parallel and equal in length, the quadrilateral is a parallelogram. Alternatively, we could show that $\overrightarrow{BC} = \overrightarrow{AD}$, or $\overrightarrow{BC} \parallel \overrightarrow{AD}$.

84. $\overrightarrow{AB} = \langle 3-(-2),\ 0-(-1)\rangle = \langle 5,\ 1\rangle = 5\mathbf{i} + \mathbf{j}$
$\overrightarrow{CD} = \langle -4-1,\ -3-(-2)\rangle = \langle -5,\ -1\rangle = -5\mathbf{i} - \mathbf{j}$
Thus, $\overrightarrow{AB} = -\overrightarrow{CD}$, so $\overrightarrow{AB} \parallel -\overrightarrow{CD}$.
$\|\overrightarrow{AB}\| = \sqrt{5^2 + 1^2} = \sqrt{26}$
$\|\overrightarrow{CD}\| = \sqrt{(-5)^2 + (-1)^2} = \sqrt{26}$
From geometry, we know that if a quadrilateral has two opposite sides that are parallel and equal in length, the quadrilateral is a parallelogram. Alternatively, we could show that $\overrightarrow{BC} = \overrightarrow{AD}$, or $\overrightarrow{BC} \parallel \overrightarrow{AD}$.

85. $\overrightarrow{PQ} = \langle 2-0, 1-(-3)\rangle = \langle 2,\ 4\rangle = 2\mathbf{i} + 4\mathbf{j}$
$\overrightarrow{PR} = \langle 3-0,\ 3-(-3)\rangle = \langle 3,\ 6\rangle = 3\mathbf{i} + 6\mathbf{j}$
$\overrightarrow{PQ} = \frac{2}{3}\overrightarrow{PR}$, so P, Q, and R are collinear.

86. $\overrightarrow{PQ} = \langle 2-(-2), 0-(-6)\rangle = \langle 4,\ 6\rangle = 4\mathbf{i} + 6\mathbf{j}$
$\overrightarrow{PR} = \langle 4-(-2),\ 3-(-6)\rangle = \langle 6,\ 9\rangle = 6\mathbf{i} + 9\mathbf{j}$
$\overrightarrow{PQ} = \frac{2}{3}\overrightarrow{PR}$, so P, Q, and R are collinear.

87. $\mathbf{v} = \mathbf{i} + 2\mathbf{j}, \mathbf{w} = \mathbf{i} - \mathbf{j}$. The lengths of the diagonals are $\|\mathbf{v} + \mathbf{w}\| = \|2\mathbf{i} + \mathbf{j}\| = \sqrt{5}$ and $\|\mathbf{v} - \mathbf{w}\| = \|3\mathbf{j}\| = 3$.

88. Let (x, y) be the fourth vertex. There are three possibilities:
1. (0, 0) and (x, y) form a diagonal
$\Rightarrow \langle 2,3\rangle = \langle x-6, y-4\rangle \Rightarrow x = 8, y = 7$;
2. (0, 0) and (2, 3) form a diagonal
$\Rightarrow \langle x, y\rangle = \langle 2-6, 3-4\rangle \Rightarrow x = -4, y = -1$;
3. (0, 0) and (6, 4) form a diagonal
$\Rightarrow \langle x, y\rangle = \langle 6-2, 4-3\rangle \Rightarrow x = 4, y = 1$.
The possibilities for the fourth vertex are (4, 1), (−4, −1), and (8, 7).

7.4 Critical Thinking/Discussion/Writing

89. $\overrightarrow{AB} = \langle 4-1, 3-2\rangle = \langle 3,1\rangle$
$\overrightarrow{CP} = \langle 3,1\rangle = \langle P_x - 6, P_y - 1\rangle$
$P_x - 6 = 3 \Rightarrow P_x = 9$
$P_y - 1 = 1 \Rightarrow P_y = 2$
The coordinates of *P* are (9, 2).

90. $\overrightarrow{CB} = \langle 4-6,\ 3-1\rangle = \langle -2,\rangle$
$\overrightarrow{PA} = \langle -2,2\rangle = \langle 1-P_x,\ 2-P_y\rangle$
$1 - P_x = -2 \Rightarrow P_x = 3$
$2 - P_y = 2 \Rightarrow P_y = 0$
The coordinates of *P* are (3, 0).

91. $\overrightarrow{CB} = \langle 4-6,\ 3-1\rangle = \langle -2,2\rangle$
$\overrightarrow{AP} = \langle -2,2\rangle = \langle P_x - 1,\ P_y - 2\rangle$
$P_x - 1 = -2 \Rightarrow P_x = -1$
$P_y - 2 = 2 \Rightarrow P_y = 4$
The coordinates of *P* are (−1, 4).

92. $\overrightarrow{BC} = \langle 6-4,\ 1-3\rangle = \langle 2,-2\rangle$
$\overrightarrow{PA} = \langle 2,\ -2\rangle = \langle 1-P_x,\ 2-P_y\rangle$
$1 - P_x = 2 \Rightarrow P_x = -1$
$2 - P_y = -2 \Rightarrow P_y = 4$
The coordinates of *P* are (−1, 4).

93. Equivalent vectors have the same length and the same direction. Therefore, two equivalent vectors with the same initial point, by definition, must have the same terminal point.

94. Answers will vary. Sample answer: Think of two equal weights hanging from either end of a horizontal hanging rod.

7.4 Maintaining Skills

95. False. If $\cos\theta > 0$, then θ lies in quadrant I or in quadrant IV.

96. False. If $\cos\theta = 0$, then
$\theta = \frac{\pi}{2} + 2\pi n$ or $\theta = \frac{3\pi}{2} + 2\pi n$, where *n* is an integer.

97. $\cos^{-1}(-1) = \pi,\ \cos^{-1}(0) = \frac{\pi}{2},\ \cos^{-1}(1) = 0$

98. $2x + 3y = 6 \Rightarrow 3y = -2x + 6 \Rightarrow y = -\frac{2}{3}x + 2$
$3x - 2y = 12 \Rightarrow -2y = -3x + 12 \Rightarrow y = \frac{3}{2}x - 6$
The slope of the first line is $-\frac{2}{3}$ and the slope of the second line is $\frac{3}{2}$. Since the product of the slopes is −1, the lines are perpendicular.

For exercises 99−102, $\mathbf{v} = \langle 1,\ -2\rangle$ and $\mathbf{w} = \langle 3,\ 2\rangle$.

99. $\|\mathbf{v}\| = \sqrt{1^2 + (-2)^2} = \sqrt{5}$

100. $\|\mathbf{w}\| = \sqrt{3^2 + 2^2} = \sqrt{13}$

101. $\frac{(1)(3) + (-2)(2)}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{(1)(3) + (-2)(2)}{\sqrt{5}\sqrt{13}} = -\frac{1}{\sqrt{65}}$

102.
$$\|\mathbf{v} + \mathbf{w}\|^2 = \|\langle 1,\ -2\rangle + \langle 3,\ 2\rangle\|^2 = \|\langle 4,\ 0\rangle\|^2 = \left(\sqrt{4^2 + 0^2}\right)^2 = 16$$
$$\begin{aligned}&\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 + 2[(1)(3) + (-2)(2)]\\ &= \|\langle 1,\ -2\rangle\|^2 + \|\langle 3,\ 2\rangle\|^2 + 2[(1)(3) + (-2)(2)]\\ &= \left(1^2 + (-2)^2\right) + \left(3^2 + 2^2\right) + 2(-1)\\ &= 5 + 13 + 2 = 16\end{aligned}$$
Thus,
$\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 + 2[(1)(3) + (-2)(2)]$.

7.5 The Dot Product

7.5 Practice Problems

1. a. $\mathbf{v} \bullet \mathbf{w} = 1(-2) + (2)(5) = 8$

b. $\mathbf{v} \bullet \mathbf{w} = 4(-3) + (-3)(-4) = 0$

2. $\mathbf{v} \bullet \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos\theta = 4(7)\cos 60° = 14$

3. There is no scalar *c* such that **v** = *c***w**, so the vectors are not parallel. Note that
$\cos\theta = \frac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{(-1)(2) + (-2)(1)}{\sqrt{5}\sqrt{5}} = -\frac{4}{5} \Rightarrow$
$\theta \ne 0$ and $\theta \ne \pi$

4. $\cos\theta = \frac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{(2)(3)+(-3)(5)}{\sqrt{2^2+(-3)^2}\sqrt{3^2+5^2}}$
$= -\frac{9}{\sqrt{13}\sqrt{34}} = -\frac{9}{\sqrt{442}} \Rightarrow$
$\theta = \cos^{-1}\left(-\frac{9}{\sqrt{442}}\right) \approx 115.3°$

5. $\|\mathbf{v}+2\mathbf{w}\|^2 = (\mathbf{v}+2\mathbf{w})\bullet(\mathbf{v}+2\mathbf{w})$
$= (\mathbf{v}+2\mathbf{w})\bullet\mathbf{v} + (\mathbf{v}+2\mathbf{w})\bullet(2\mathbf{w})$
$= \mathbf{v}\bullet\mathbf{v} + 2\mathbf{w}\bullet\mathbf{v} + \mathbf{v}\bullet 2\mathbf{w} + (2\mathbf{w})\bullet(2\mathbf{w})$
$= \|\mathbf{v}\|^2 + 4(\mathbf{v}\bullet\mathbf{w}) + 4\|\mathbf{w}\|^2$
$= \|\mathbf{v}\|^2 + 4\|\mathbf{v}\|\|\mathbf{w}\|\cos\theta + 4\|\mathbf{w}\|^2$
$= 20^2 + 4(20)(12)\cos 54° + 4(12)^2$
≈ 1540.2738
$\|\mathbf{v}+2\mathbf{w}\| \approx 39$

6. $\mathbf{v}\bullet\mathbf{w} = 4(2)+(8)(-1) = 0$
The vectors are orthogonal.

7. The vectors are orthogonal if the dot product equals zero.
$\mathbf{v}\bullet\mathbf{w} = 4k + (-2)(6) = 0$
$4k - 12 = 0 \Rightarrow k = 3$

8. First compute $\|\mathbf{v}\|, \|\mathbf{w}\|$, and $\mathbf{v}\bullet\mathbf{w}$.
$\|\mathbf{v}\| = \sqrt{1^2+(-2)^2} = \sqrt{5}$
$\|\mathbf{w}\| = \sqrt{2^2+5^2} = \sqrt{29}$
$\mathbf{v}\bullet\mathbf{w} = 1(2)+(-2)(5) = -8$
The vector projection of **v** onto **w** is
$\text{proj}_{\mathbf{w}}\,\mathbf{v} = \frac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w} = -\frac{8}{29}\langle 2,5\rangle = \left\langle -\frac{16}{29}, -\frac{40}{29}\right\rangle.$

The scalar projection of **v** onto **w** is
$\frac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{w}\|} = -\frac{8}{\sqrt{29}} = -\frac{8\sqrt{29}}{29}.$

9. First compute $\mathbf{v}\bullet\mathbf{w}$ and $\|\mathbf{w}\|^2$.
$\mathbf{v}\bullet\mathbf{w} = 5(4)+1(4) = 24$
$\|\mathbf{w}\|^2 = 4^2+4^2 = 32$
$\mathbf{v_1} = \text{proj}_{\mathbf{w}}\,\mathbf{v} = \left(\frac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{w}\|^2}\right)\mathbf{w} = \frac{24}{32}\langle 4,4\rangle = \langle 3,3\rangle$
$\mathbf{v_2} = \mathbf{v} - \mathbf{v_1} = \langle 5,1\rangle - \langle 3,3\rangle = \langle 2,-2\rangle$
Check that $\mathbf{v_2}$ is orthogonal to **w** by showing that $\mathbf{v_2}\bullet\mathbf{w} = 0$: $2(4)+(-2)(4) = 0.$

Thus
$\mathbf{v} = \mathbf{v_1} + \mathbf{v_2}$
$\langle 5,1\rangle = \langle 3,3\rangle + \langle 2,-2\rangle$

10. $W = \mathbf{F}\bullet\overrightarrow{PQ} = \|\mathbf{F}\|\left\|\overrightarrow{PQ}\right\|\cos\theta$
$= 40(150)\cos 60° = 3000$ foot-pounds

7.5 The Dot Product

1. The dot product of $\mathbf{v} = \langle a_1, a_2\rangle$ and $\mathbf{w} = \langle b_1, b_2\rangle$ is defined as $\mathbf{v}\bullet\mathbf{w} = \underline{a_1b_1 + a_2b_2}$.

2. If θ is the angle between the vectors **v** and **w**, then $\mathbf{v}\bullet\mathbf{w} = \underline{\|\mathbf{v}\|\|\mathbf{w}\|\cos\theta}$.

3. If **v** and **w** are orthogonal, then $\mathbf{v}\bullet\mathbf{w} = \underline{0}$.

4. If **v** and **w** are nonzero vectors with $\mathbf{v}\bullet\mathbf{w} = 0$, then **v** and **w** are <u>orthogonal</u>.

5. True. If $\mathbf{v}\bullet\mathbf{w} < 0$, then $\cos\theta$ is negative, and θ is obtuse.

6. True

7. $\mathbf{u}\bullet\mathbf{v} = 1(3)+(-2)(5) = -7$

8. $\mathbf{u}\bullet\mathbf{v} = 1(-3)+(3)(1) = 0$

9. $\mathbf{u}\bullet\mathbf{v} = 2(3)+(-6)(1) = 0$

10. $\mathbf{u}\bullet\mathbf{v} = (-1)(-3)+(-2)(-4) = 11$

11. $\mathbf{u}\bullet\mathbf{v} = 1(8)+(-3)(-2) = 14$

12. $\mathbf{u}\bullet\mathbf{v} = 6(2)+(-1)(7) = 5$

13. $\mathbf{u}\bullet\mathbf{v} = 4(0)+(-2)(3) = -6$

14. $\mathbf{u}\bullet\mathbf{v} = (-2)(0)+(0)(5) = 0$

15. $\mathbf{u}\bullet\mathbf{v} = 2(5)\cos\frac{\pi}{6} = 5\sqrt{3} \approx 8.7$

16. $\mathbf{u}\bullet\mathbf{v} = 3(4)\cos\frac{\pi}{3} = 6$

17. $\mathbf{u}\bullet\mathbf{v} = 5(4)\cos 65° \approx 8.5$

18. $\mathbf{u}\bullet\mathbf{v} = 5(3)\cos 78° \approx 3.1$

19. $\mathbf{u}\bullet\mathbf{v} = 3(7)\cos 120° = -10.5$

20. $\mathbf{u}\bullet\mathbf{v} = 6(4)\cos 150° \approx -20.8$

21. $\cos\theta = \dfrac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \dfrac{\sqrt{3}}{2\sqrt{3}} = \dfrac{1}{2} \Rightarrow$
$\theta = \cos^{-1}\left(\dfrac{1}{2}\right) = 60°$

22. $\cos\theta = \dfrac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \dfrac{-\sqrt{8}}{1\cdot 4} = -\dfrac{2\sqrt{2}}{4} = -\dfrac{\sqrt{2}}{2} \Rightarrow$
$\theta = \cos^{-1}\left(-\dfrac{\sqrt{2}}{2}\right) = 135°$

23. $\cos\theta = \dfrac{(-2)(1)+(-3)(1)}{\sqrt{(-2)^2+(-3)^2}\sqrt{1^2+1^2}} = -\dfrac{5}{\sqrt{26}} \Rightarrow$
$\theta = \cos^{-1}\left(-\dfrac{5}{\sqrt{26}}\right) \approx 168.7°$

24. $\cos\theta = \dfrac{(1)(3)+(-1)(-4)}{\sqrt{1^2+(-1)^2}\sqrt{3^2+4^2}} = \dfrac{7}{5\sqrt{2}} \Rightarrow$
$\theta = \cos^{-1}\left(\dfrac{7}{5\sqrt{2}}\right) \approx 8.1°$

25. $\cos\theta = \dfrac{2(-5)+5(2)}{\sqrt{2^2+5^2}\sqrt{(-5)^2+2^2}} = 0 \Rightarrow$
$\theta = \cos^{-1}(0) = 90°$

26. $\cos\theta = \dfrac{3(7)+(-7)(3)}{\sqrt{3^2+(-7)^2}\sqrt{7^2+3^2}} = 0 \Rightarrow$
$\theta = \cos^{-1}(0) = 90°$

27. $\cos\theta = \dfrac{1(3)+1(3)}{\sqrt{1^2+1^2}\sqrt{3^2+3^2}} = 1 \Rightarrow$
$\theta = \cos^{-1}(1) = 0°$

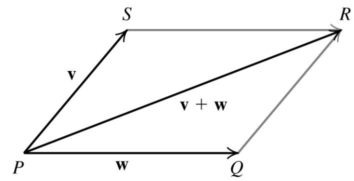
28. $\cos\theta = \dfrac{(-2)(-4)+3(6)}{\sqrt{(-2)^2+3^2}\sqrt{(-4)^2+6^2}} = 1 \Rightarrow$
$\theta = \cos^{-1}(1) = 0°$

29. $\cos\theta = \dfrac{(2)(-4)+(-3)(6)}{\sqrt{(2)^2+(-3)^2}\sqrt{(-4)^2+(6)^2}}$
$= -\dfrac{26}{26} = -1 \Rightarrow \theta = \cos^{-1}(-1) = 180°$

30. $\cos\theta = \dfrac{(-3)(6)+(4)(-8)}{\sqrt{(-3)^2+4^2}\sqrt{6^2+(-8)^2}} = -1 \Rightarrow$
$\theta = \cos^{-1}(-1) = 180°$

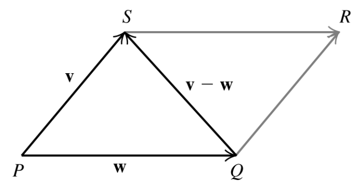
31. Using the Law of Cosines, we have

$$\begin{aligned}\|\mathbf{v}+\mathbf{w}\|^2 &= \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\|\mathbf{v}\|\|\mathbf{w}\|\cos\theta \\ &= 4^2 + 5^2 - 2(4)(5)\cos(180° - 60°) \\ &= 41 - 40\cos 120° = 41 - 40\left(-\frac{1}{2}\right) \\ &= 61 \Rightarrow \|\mathbf{v}+\mathbf{w}\| = \sqrt{61}\end{aligned}$$



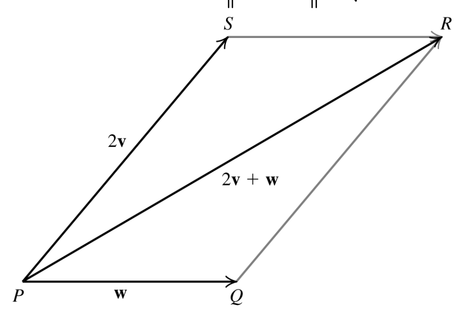
32. Using the Law of Cosines, we have

$$\begin{aligned}\|\mathbf{v}-\mathbf{w}\|^2 &= \|\mathbf{v}\|^2 + \|-\mathbf{w}\|^2 - 2\|\mathbf{v}\|\|-\mathbf{w}\|\cos\theta \\ &= 4^2 + (-5)^2 - 2(4)(5)\cos 60° \\ &= 41 - 40\cos 60° = 41 - 40\left(\frac{1}{2}\right) \\ &= 21 \Rightarrow \|\mathbf{v}-\mathbf{w}\| = \sqrt{21}\end{aligned}$$



33. Using the Law of Cosines, we have

$$\begin{aligned}\|2\mathbf{v}+\mathbf{w}\|^2 &= \|2\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\|2\mathbf{v}\|\|\mathbf{w}\|\cos\theta \\ &= ((2)(4))^2 + 5^2 \\ &\quad - 2(2)(4)(5)\cos(180° - 60°) \\ &= 89 - 80\cos 120° = 89 - 80\left(-\frac{1}{2}\right) \\ &= 129 \Rightarrow \|2\mathbf{v}+\mathbf{w}\| = \sqrt{129}\end{aligned}$$



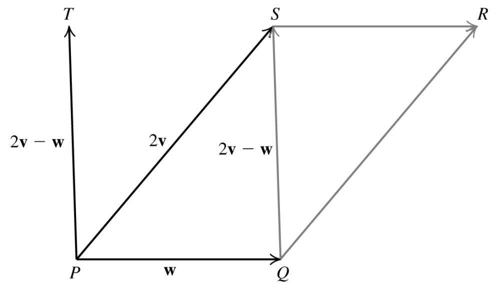
34. Using the Law of Cosines, we have

$$\begin{aligned}\|2\mathbf{v}-\mathbf{w}\|^2 &= \|2\mathbf{v}\|^2 + \|-\mathbf{w}\|^2 - 2\|2\mathbf{v}\|\|-\mathbf{w}\|\cos\theta \\ &= ((2)(4))^2 + 5^2 \\ &\quad - 2(2)(4)(5)\cos(60°) \\ &= 89 - 80\cos 60° = 89 - 80\left(\frac{1}{2}\right) = 49\end{aligned}$$

$\|2\mathbf{v}-\mathbf{w}\| = 7$

(*continued on next page*)

(*continued*)



For exercises 35−38, we have

$$\mathbf{v} = 4\cos 60°\mathbf{i} + 4\sin 60°\mathbf{j} = 4\left(\frac{1}{2}\right)\mathbf{i} + 4\left(\frac{\sqrt{3}}{2}\right)\mathbf{j} = 2\mathbf{i} + 2\sqrt{3}\mathbf{j}$$

$$\mathbf{w} = 5\cos 0°\mathbf{i} + 5\sin 0°\mathbf{j} = 5\mathbf{i}$$

35. Using the figure and the result from exercise 31, the angle we want is $\angle RPQ$.

$$\cos\theta = \frac{(\mathbf{v}+\mathbf{w})\bullet\mathbf{w}}{\|\mathbf{v}+\mathbf{w}\|\|\mathbf{w}\|} = \frac{\langle 7, 2\sqrt{3}\rangle\bullet\langle 5, 0\rangle}{5\sqrt{61}} = \frac{7(5)+(2\sqrt{3})(0)}{5\sqrt{61}} = \frac{7}{\sqrt{61}} \Rightarrow$$

$$\theta = \cos^{-1}\left(\frac{7}{\sqrt{61}}\right)$$

36. Using the figure and the result from exercise 32, the angle we want is $\angle SQR$. So

$$\cos\theta = \frac{(\mathbf{v}-\mathbf{w})\bullet\mathbf{v}}{\|\mathbf{v}-\mathbf{w}\|\|\mathbf{v}\|} = \frac{\langle -3, 2\sqrt{3}\rangle\bullet\langle 2, 2\sqrt{3}\rangle}{4\sqrt{21}} = \frac{-3(2)+(2\sqrt{3})(2\sqrt{3})}{4\sqrt{21}} = \frac{-6+12}{4\sqrt{21}} = \frac{3}{2\sqrt{21}} \Rightarrow \theta = \cos^{-1}\left(\frac{3}{2\sqrt{21}}\right)$$

37. Using the figure and the result from exercise 33, the angle we want is $\angle SPR$. So

$$2\mathbf{v}+\mathbf{w} = 2\langle 2, 2\sqrt{3}\rangle + \langle 5, 0\rangle = \langle 9, 4\sqrt{3}\rangle$$

$$\cos\theta = \frac{(2\mathbf{v}+\mathbf{w})\bullet\mathbf{v}}{\|2\mathbf{v}+\mathbf{w}\|\|\mathbf{v}\|} = \frac{\langle 9, 4\sqrt{3}\rangle\bullet\langle 2, 2\sqrt{3}\rangle}{4\sqrt{129}} = \frac{9(2)+(4\sqrt{3})(2\sqrt{3})}{4\sqrt{129}} = \frac{18+24}{4\sqrt{21}} = \frac{21}{2\sqrt{129}} \Rightarrow \theta = \cos^{-1}\left(\frac{21}{2\sqrt{129}}\right)$$

38. Using the figure and the result from exercise 34, the angle we want is $\angle TPQ$. So

$$2\mathbf{v}-\mathbf{w} = 2\langle 2, 2\sqrt{3}\rangle + \langle 5, 0\rangle = \langle -1, 4\sqrt{3}\rangle$$

$$\cos\theta = \frac{(2\mathbf{v}-\mathbf{w})\bullet\mathbf{w}}{\|2\mathbf{v}-\mathbf{w}\|\|\mathbf{w}\|} = \frac{\langle -1, 4\sqrt{3}\rangle\bullet\langle 5, 0\rangle}{5\cdot 7} = \frac{-5}{35} = -\frac{1}{7} \Rightarrow \theta = \cos^{-1}\left(-\frac{1}{7}\right)$$

In exercises 39−44, use the fact that the dot product of vectors that are orthogonal equals zero.

39. $\mathbf{v}\bullet\mathbf{w} = 2(-3)+3(2) = 0$
The vectors are orthogonal.

40. $\mathbf{v}\bullet\mathbf{w} = -4(5)+(-5)(-4) = 0$
The vectors are orthogonal.

41. $\mathbf{v}\bullet\mathbf{w} = 2(-7)+7(-2) = -28$
The vectors are not orthogonal.

42. $\mathbf{v}\bullet\mathbf{w} = -3(4)+4(-3) = -24$
The vectors are not orthogonal.

43. $\mathbf{v}\bullet\mathbf{w} = 12(2)+6(-4) = 0$
The vectors are orthogonal.

44. $\mathbf{v}\bullet\mathbf{w} = -9(4)+6(6) = 0$
The vectors are orthogonal.

In exercises 45−50, use the fact that the angle between two parallel vectors is either 0 or π. Also, two vectors **v** and **w** are parallel if there is a nonzero scalar c such that $\mathbf{v} = c\mathbf{w}$.

45. $$\cos\theta = \frac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{(2)\left(\frac{2}{3}\right)+(3)(1)}{\sqrt{2^2+3^2}\sqrt{\left(\frac{2}{3}\right)^2+1^2}} = \frac{\frac{13}{3}}{\sqrt{13}\sqrt{\frac{13}{9}}} = 1 \Rightarrow \theta = 0$$

The vectors are parallel.
Alternatively, notice that $\mathbf{v} = 3\mathbf{w}$.

46. $$\cos\theta = \frac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{(5)(-10)+(-2)(4)}{\sqrt{5^2+(-2)^2}\sqrt{(-10)^2+(4)^2}} = -\frac{58}{\sqrt{29}\sqrt{116}} = -\frac{58}{\sqrt{3364}} = -\frac{58}{58} \Rightarrow$$

$\theta = \pi$
The vectors are parallel.

47. $\cos\theta = \dfrac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \dfrac{(3)(-6)+(5)(10)}{\sqrt{3^2+5^2}\sqrt{(-6)^2+10^2}}$
$= \dfrac{32}{\sqrt{34}\sqrt{136}} \Rightarrow \theta \neq 0° \text{ or } \theta \neq \pi$
The vectors are not parallel.

48. $\cos\theta = \dfrac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \dfrac{(-4)(-1)+(-2)\left(\frac{1}{2}\right)}{\sqrt{(-4)^2+(-2)^2}\sqrt{(-1)^2+\left(\frac{1}{2}\right)^2}}$
$= \dfrac{3}{\sqrt{20}\sqrt{\frac{5}{4}}} = \dfrac{3}{5} \Rightarrow \theta \neq 0° \text{ or } \theta \neq \pi$
The vectors are not parallel.

49. $\cos\theta = \dfrac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \dfrac{(6)(2)+(3)(1)}{\sqrt{6^2+3^2}\sqrt{2^2+1^2}}$
$= \dfrac{15}{\sqrt{45}\sqrt{5}} = \dfrac{15}{\sqrt{225}} = \dfrac{15}{15} = 1 \Rightarrow \theta = 0°$
The vectors are parallel.
Alternatively, notice that $\mathbf{v} = \dfrac{1}{3}\mathbf{w}$.

50. $\cos\theta = \dfrac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \dfrac{(2)(-4)+(5)(-10)}{\sqrt{2^2+5^2}\sqrt{(-4)^2+(-10)^2}}$
$= -\dfrac{58}{\sqrt{29}\sqrt{116}} = -\dfrac{58}{\sqrt{3364}} = -\dfrac{58}{58} = 1 \Rightarrow$
$\theta = 0°$
The vectors are parallel.

In exercises 51−54, use the facts that two vectors are orthogonal if the dot product equals zero, and two vectors **v** and **w** are parallel if there is a nonzero scalar *n* such that **v** = *n***w**.

51. a. $\mathbf{v}\bullet\mathbf{w} = 2(3)+3c = 0$
$6+3c = 0 \Rightarrow c = -2$

b. $\mathbf{v} = n\mathbf{w}$
$2 = 3n \Rightarrow \dfrac{2}{3} = n$
$\langle 2,3\rangle = \dfrac{2}{3}\langle 3,c\rangle \Rightarrow 3 = \dfrac{2}{3}c \Rightarrow c = \dfrac{9}{2}$

52. a. $\mathbf{v}\bullet\mathbf{w} = 1(c)+(-2)(4) = 0$
$c-8 = 0 \Rightarrow c = 8$

b. $\mathbf{v} = n\mathbf{w}$
$-2 = 4n \Rightarrow -\dfrac{1}{2} = n$
$\langle 1,-2\rangle = -\dfrac{1}{2}\langle c,4\rangle \Rightarrow 1 = -\dfrac{1}{2}c \Rightarrow c = -2$

53. a. $\mathbf{v}\bullet\mathbf{w} = -4c+3(-6) = 0$
$-4c-18 = 0 \Rightarrow c = -\dfrac{9}{2}$

b. $\mathbf{v} = n\mathbf{w}$
$3 = -6n \Rightarrow -\dfrac{1}{2} = n$
$\langle -4,3\rangle = -\dfrac{1}{2}\langle c,-6\rangle \Rightarrow -4 = -\dfrac{1}{2}c \Rightarrow c = 8$

54. a. $\mathbf{v}\bullet\mathbf{w} = -1(-2)+(-3)c = 0$
$2-3c = 0 \Rightarrow c = \dfrac{2}{3}$

b. $\mathbf{v} = n\mathbf{w}$
$-1 = -2n \Rightarrow \dfrac{1}{2} = n$
$\langle -1,-3\rangle = \dfrac{1}{2}\langle -2,c\rangle \Rightarrow -3 = \dfrac{1}{2}c \Rightarrow c = -6$

55. $\text{proj}_{\mathbf{w}}\,\mathbf{v} = \dfrac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w} = \dfrac{3(4)+5(1)}{4^2+1^2}\langle 4,1\rangle$
$= \dfrac{17}{17}\langle 4,1\rangle = \langle 4,1\rangle$

56. $\text{proj}_{\mathbf{w}}\,\mathbf{v} = \dfrac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w} = \dfrac{3(-4)+5(2)}{(-4)^2+2^2}\langle -4,2\rangle$
$= -\dfrac{2}{20}\langle -4,2\rangle = \left\langle \dfrac{2}{5},-\dfrac{1}{5}\right\rangle$

57. $\text{proj}_{\mathbf{w}}\,\mathbf{v} = \dfrac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w} = \dfrac{0(3)+2(4)}{3^2+4^2}\langle 3,4\rangle$
$= \dfrac{8}{25}\langle 3,4\rangle = \left\langle \dfrac{24}{25},\dfrac{32}{25}\right\rangle$

58. $\text{proj}_{\mathbf{w}}\,\mathbf{v} = \dfrac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w} = \dfrac{2(4)+0(-3)}{4^2+(-3)^2}\langle 4,-3\rangle$
$= \dfrac{8}{25}\langle 4,-3\rangle = \left\langle \dfrac{32}{25},-\dfrac{24}{25}\right\rangle$

59. $\text{proj}_{\mathbf{w}}\,\mathbf{v} = \dfrac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w} = \dfrac{5(1)+(-3)(0)}{1^2+0^2}\langle 1,0\rangle$
$= 5\langle 1,0\rangle = \langle 5,0\rangle$

60. $\text{proj}_{\mathbf{w}}\,\mathbf{v} = \dfrac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w} = \dfrac{2(0)+7(1)}{0^2+1^2}\langle 0,1\rangle$
$= 7\langle 0,1\rangle = \langle 0,7\rangle$

61. $\text{proj}_{\mathbf{w}}\,\mathbf{v} = \dfrac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w} = \dfrac{a(1)+b(1)}{1^2+1^2}\langle 1,1\rangle$
$= \dfrac{a+b}{2}\langle 1,1\rangle = \left\langle \dfrac{a+b}{2},\dfrac{a+b}{2}\right\rangle$

62. $\text{proj}_{\mathbf{w}}\,\mathbf{v} = \frac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w} = \frac{a(1)+b(-1)}{1^2+(-1)^2}\langle 1,-1\rangle$
$= \frac{a-b}{2}\langle 1,-1\rangle = \left\langle \frac{a-b}{2}, \frac{-a+b}{2}\right\rangle$

63. First compute $\mathbf{v}\bullet\mathbf{w}$ and $\|\mathbf{w}\|^2$.
$\mathbf{v}\bullet\mathbf{w} = 2(1)+3(0) = 2$
$\|\mathbf{w}\|^2 = 1^2+0^2 = 1$
$\mathbf{v_1} = \text{proj}_{\mathbf{w}}\,\mathbf{v} = \left(\frac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{w}\|^2}\right)\mathbf{w} = \frac{2}{1}\langle 1,0\rangle = \langle 2,0\rangle$
$\mathbf{v_2} = \mathbf{v}-\mathbf{v_1} = \langle 2,3\rangle - \langle 2,0\rangle = \langle 0,3\rangle$
Check that $\mathbf{v_2}$ is orthogonal to **w** by showing that $\mathbf{v_2}\bullet\mathbf{w} = 0: 0(1)+3(0) = 0.$
$\mathbf{v} = \mathbf{v_1}+\mathbf{v_2} \Rightarrow \langle 2,3\rangle = \langle 2,0\rangle + \langle 0,3\rangle$

64. First compute $\mathbf{v}\bullet\mathbf{w}$ and $\|\mathbf{w}\|^2$.
$\mathbf{v}\bullet\mathbf{w} = 4(0)+(-3)(1) = -3$
$\|\mathbf{w}\|^2 = 0^2+1^2 = 1$
$\mathbf{v_1} = \text{proj}_{\mathbf{w}}\,\mathbf{v} = \left(\frac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{w}\|^2}\right)\mathbf{w} = -\frac{3}{1}\langle 0,1\rangle = \langle 0,-3\rangle$
$\mathbf{v_2} = \mathbf{v}-\mathbf{v_1} = \langle 4,-3\rangle - \langle 0,-3\rangle = \langle 4,0\rangle$
Check that $\mathbf{v_2}$ is orthogonal to **w** by showing that $\mathbf{v_2}\bullet\mathbf{w} = 0: 4(0)+0(1) = 0.$
$\mathbf{v} = \mathbf{v_1}+\mathbf{v_2} \Rightarrow \langle 4,-3\rangle = \langle 0,-3\rangle + \langle 4,0\rangle$

In exercises 65–70, we do not include the check step for brevity.

65. First compute $\mathbf{v}\bullet\mathbf{w}$ and $\|\mathbf{w}\|^2$.
$\mathbf{v}\bullet\mathbf{w} = 1(1)+0(1) = 1$
$\|\mathbf{w}\|^2 = 1^2+1^2 = 2$
$\mathbf{v_1} = \text{proj}_{\mathbf{w}}\,\mathbf{v} = \left(\frac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{w}\|^2}\right)\mathbf{w} = \frac{1}{2}\langle 1,1\rangle = \left\langle \frac{1}{2},\frac{1}{2}\right\rangle$
$\mathbf{v_2} = \mathbf{v}-\mathbf{v_1} = \langle 1,0\rangle - \left\langle \frac{1}{2},\frac{1}{2}\right\rangle = \left\langle \frac{1}{2},-\frac{1}{2}\right\rangle$
$\mathbf{v} = \mathbf{v_1}+\mathbf{v_2} \Rightarrow \langle 1,0\rangle = \left\langle \frac{1}{2},\frac{1}{2}\right\rangle + \left\langle \frac{1}{2},-\frac{1}{2}\right\rangle$

66. First compute $\mathbf{v}\bullet\mathbf{w}$ and $\|\mathbf{w}\|^2$.
$\mathbf{v}\bullet\mathbf{w} = 0(4)+1(-3) = -3$
$\|\mathbf{w}\|^2 = 4^2+(-3)^2 = 25$
$\mathbf{v_1} = \text{proj}_{\mathbf{w}}\,\mathbf{v} = \left(\frac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{w}\|^2}\right)\mathbf{w} = -\frac{3}{25}\langle 4,-3\rangle$
$= \left\langle -\frac{12}{25},\frac{9}{25}\right\rangle$
$\mathbf{v_2} = \mathbf{v}-\mathbf{v_1} = \langle 0,1\rangle - \left\langle -\frac{12}{25},\frac{9}{25}\right\rangle = \left\langle \frac{12}{25},\frac{16}{25}\right\rangle$
$\mathbf{v} = \mathbf{v_1}+\mathbf{v_2} \Rightarrow \langle 0,1\rangle = \left\langle -\frac{12}{25},\frac{9}{25}\right\rangle + \left\langle \frac{12}{25},\frac{16}{25}\right\rangle$

67. First compute $\mathbf{v}\bullet\mathbf{w}$ and $\|\mathbf{w}\|^2$.
$\mathbf{v}\bullet\mathbf{w} = 4(-2)+2(4) = 0$
$\|\mathbf{w}\|^2 = (-2)^2+4^2 = 20$
$\mathbf{v_1} = \text{proj}_{\mathbf{w}}\,\mathbf{v} = \left(\frac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{w}\|^2}\right)\mathbf{w} = \frac{0}{20}\langle -2,4\rangle = \langle 0,0\rangle$
$\mathbf{v_2} = \mathbf{v}-\mathbf{v_1} = \langle 4,2\rangle - \langle 0,0\rangle = \langle 4,2\rangle$
$\mathbf{v} = \mathbf{v_1}+\mathbf{v_2} \Rightarrow \langle 4,2\rangle = \langle 0,0\rangle + \langle 4,2\rangle$

68. First compute $\mathbf{v}\bullet\mathbf{w}$ and $\|\mathbf{w}\|^2$.
$\mathbf{v}\bullet\mathbf{w} = 5(3)+2(5) = 25$
$\|\mathbf{w}\|^2 = 3^2+5^2 = 34$
$\mathbf{v_1} = \text{proj}_{\mathbf{w}}\,\mathbf{v} = \left(\frac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{w}\|^2}\right)\mathbf{w} = \frac{25}{34}\langle 3,5\rangle$
$= \left\langle \frac{75}{34},\frac{125}{34}\right\rangle$
$\mathbf{v_2} = \mathbf{v}-\mathbf{v_1} = \langle 5,2\rangle - \left\langle \frac{75}{34},\frac{125}{34}\right\rangle = \left\langle \frac{95}{34},-\frac{57}{34}\right\rangle$
$\mathbf{v} = \mathbf{v_1}+\mathbf{v_2} \Rightarrow \langle 5,2\rangle = \left\langle \frac{75}{34},\frac{125}{34}\right\rangle + \left\langle \frac{95}{34},-\frac{57}{34}\right\rangle$

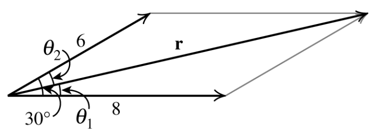
69. First compute $\mathbf{v}\bullet\mathbf{w}$ and $\|\mathbf{w}\|^2$.
$\mathbf{v}\bullet\mathbf{w} = 4(3)+(-1)(4) = 8$
$\|\mathbf{w}\|^2 = 3^2+4^2 = 25$
$\mathbf{v_1} = \text{proj}_{\mathbf{w}}\,\mathbf{v} = \left(\frac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{w}\|^2}\right)\mathbf{w} = \frac{8}{25}\langle 3,4\rangle$
$= \left\langle \frac{24}{25},\frac{32}{25}\right\rangle$
$\mathbf{v_2} = \mathbf{v}-\mathbf{v_1} = \langle 4,-1\rangle - \left\langle \frac{24}{25},\frac{32}{25}\right\rangle = \left\langle \frac{76}{25},-\frac{57}{25}\right\rangle$
$\mathbf{v} = \mathbf{v_1}+\mathbf{v_2} \Rightarrow \langle 4,-1\rangle = \left\langle \frac{24}{25},\frac{32}{25}\right\rangle + \left\langle \frac{76}{25},-\frac{57}{25}\right\rangle$

70. First compute $\mathbf{v} \cdot \mathbf{w}$ and $\|\mathbf{w}\|^2$.

$$\mathbf{v} \cdot \mathbf{w} = (-3)(3) + 5(-4) = -29$$
$$\|\mathbf{w}\|^2 = 3^2 + (-4)^2 = 25$$
$$\mathbf{v_1} = \text{proj}_{\mathbf{w}}\ \mathbf{v} = \left(\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|^2}\right)\mathbf{w} = -\frac{29}{25}\langle 3, -4\rangle = \left\langle -\frac{87}{25}, \frac{116}{25}\right\rangle$$
$$\mathbf{v_2} = \mathbf{v} - \mathbf{v_1} = \langle -3, 5\rangle - \left\langle -\frac{87}{25}, \frac{116}{25}\right\rangle = \left\langle \frac{12}{25}, \frac{9}{25}\right\rangle$$
$$\mathbf{v} = \mathbf{v_1} + \mathbf{v_2} \Rightarrow \langle -3, 5\rangle = \left\langle -\frac{87}{25}, \frac{116}{25}\right\rangle + \left\langle \frac{12}{25}, \frac{9}{25}\right\rangle$$

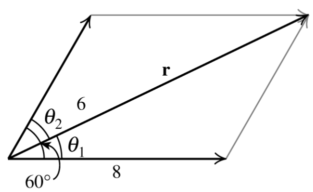
7.5 Applying the Concepts

71.



$$\mathbf{v} = 8\cos 0°\mathbf{i} + 8\sin 0°\mathbf{j} = 8\mathbf{i}$$
$$\mathbf{w} = 6\cos 30°\mathbf{i} + 6\sin 30°\mathbf{j} = 3\sqrt{3}\mathbf{i} + 3\mathbf{j}$$
$$\mathbf{r} = \mathbf{v} + \mathbf{w} = \left(8 + 3\sqrt{3}\right)\mathbf{i} + 3\mathbf{j}$$
$$\|\mathbf{r}\| = \sqrt{\left(8 + 3\sqrt{3}\right)^2 + 3^2} \approx 13.53 \approx 13.5 \text{ pounds}$$
$$\cos\theta_1 = \frac{\left(\left(8 + 3\sqrt{3}\right)\mathbf{i} + 3\mathbf{j}\right) \cdot 8\mathbf{i}}{(13.53)(8)} = \frac{8 + 3\sqrt{3}}{13.53} \Rightarrow \theta_1 = \cos^{-1}\left(\frac{8 + 3\sqrt{3}}{13.53}\right) \approx 12.8°$$
$$\theta_2 \approx 30° - 12.8° \approx 17.2°$$

72.



$$\mathbf{v} = 8\cos 0°\mathbf{i} + 8\sin 0°\mathbf{j} = 8\mathbf{i}$$
$$\mathbf{w} = 6\cos 60°\mathbf{i} + 6\sin 60°\mathbf{j} = 3\mathbf{i} + 3\sqrt{3}\mathbf{j}$$
$$\mathbf{r} = \mathbf{v} + \mathbf{w} = 11\mathbf{i} + 3\sqrt{3}\mathbf{j}$$
$$\|\mathbf{r}\| = \sqrt{11^2 + \left(3\sqrt{3}\right)^2} \approx 12.17 \approx 12.2 \text{ pounds}$$
$$\cos\theta_1 = \frac{\left(11\mathbf{i} + 3\sqrt{3}\mathbf{j}\right) \cdot 8\mathbf{i}}{(12.17)(8)} = \frac{11}{12.17} \Rightarrow \theta_1 = \cos^{-1}\left(\frac{11}{12.17}\right) \approx 25.3°;\ \ \theta_2 \approx 60° - 25.3° \approx 34.7°$$

73. $\mathbf{F} = \langle 64, -20\rangle$

$$\overrightarrow{OP} = \langle 10, 4\rangle$$
$$W = \mathbf{F} \cdot \overrightarrow{OP} = (64)(10) + (-20)(4) = 560 \text{ foot-pounds}$$

74. $\mathbf{F} = 60(\cos 18°\mathbf{i} + \sin 18°\mathbf{j})$

$$\overrightarrow{OP} = \langle 150, 0\rangle$$
$$W = \mathbf{F} \cdot \overrightarrow{OP} = 60\cos 18°(150) + 60\sin 18°(0) = 8559.5 \text{ foot-pounds}$$

75. $5220 = \mathbf{F}_1 \cos 18° \approx 5520.2$ lb

76. $\overrightarrow{OP} = \langle 10, 0 \rangle$

Using the result from exercise 75, we have $W = \mathbf{F} \bullet \overrightarrow{OP} = 5520.2(10) + \mathbf{F}_2(0) = 55,202$ foot-pounds

77. $\mathbf{F} = \left\langle 10\cos\frac{2\pi}{3}, 10\sin\frac{2\pi}{3} \right\rangle;\ \overrightarrow{OP} = \langle -6, 3 \rangle$

$W = \mathbf{F} \bullet \overrightarrow{OP} = \left(10\cos\frac{2\pi}{3}\right)(-6) + \left(10\sin\frac{2\pi}{3}\right)(3) \approx 56$ foot-pounds

78. $\mathbf{F}_1 = 3\mathbf{i} - 2\mathbf{j},\quad \mathbf{F}_2 = -4\mathbf{i} + 3\mathbf{j};\quad \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = -\mathbf{i} + \mathbf{j} = \langle -1, 1 \rangle$

$\overrightarrow{PQ} = \langle 3 - 8, 5 - 4 \rangle = \langle -5, 1 \rangle$

$W = \mathbf{F} \bullet \overrightarrow{PQ} = (-1)(-5) + (1)(1) = 6$ foot-pounds

7.5 Beyond the Basics

79. $\overrightarrow{CA} = \langle a - x, -y \rangle,\ \overrightarrow{CB} = \langle -a - x, -y \rangle$

$\overrightarrow{CA} \bullet \overrightarrow{CB} = (a - x)(-a - x) + (-y)(-y) = -a^2 + x^2 + y^2 = -a^2 + a^2 = 0$ (using the hint)

Since the dot product equals 0, the vectors are orthogonal (perpendicular).

80. Since $|\mathbf{v} \bullet \mathbf{w}| = \|\mathbf{v}\|\|\mathbf{w}\|\cos\theta,\ |\mathbf{v} \bullet \mathbf{w}| = \|\mathbf{v}\|\|\mathbf{w}\|$ when $\cos\theta = 1$ or $\theta = 0$.

81. Given $\mathbf{u} = \langle \mathbf{u}_1, \mathbf{u}_2 \rangle,\ \mathbf{v} = \langle \mathbf{v}_1, \mathbf{v}_2 \rangle$, and $\mathbf{w} = \langle \mathbf{w}_1, \mathbf{w}_2 \rangle$. Then $(\mathbf{v} + \mathbf{w}) = \langle \mathbf{v}_1 + \mathbf{w}_1, \mathbf{v}_2 + \mathbf{w}_2 \rangle$.

$$\begin{aligned}\mathbf{u} \bullet (\mathbf{v} + \mathbf{w}) &= \mathbf{u}_1(\mathbf{v}_1 + \mathbf{w}_1) + \mathbf{u}_2(\mathbf{v}_2 + \mathbf{w}_2) = \mathbf{u}_1\mathbf{v}_1 + \mathbf{u}_1\mathbf{w}_1 + \mathbf{u}_2\mathbf{v}_2 + \mathbf{u}_2\mathbf{w}_2 = (\mathbf{u}_1\mathbf{v}_1 + \mathbf{u}_2\mathbf{v}_2) + (\mathbf{u}_1\mathbf{w}_1 + \mathbf{u}_2\mathbf{w}_2)\\ &= \mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{w}\end{aligned}$$

82. Given $\mathbf{v} = \langle \mathbf{v}_1, \mathbf{v}_2 \rangle$ and $\mathbf{w} = \langle \mathbf{w}_1, \mathbf{w}_2 \rangle$. Then $(\mathbf{v} + \mathbf{w}) = \langle \mathbf{v}_1 + \mathbf{w}_1, \mathbf{v}_2 + \mathbf{w}_2 \rangle$.

$$\begin{aligned}(\mathbf{v} + \mathbf{w}) \bullet (\mathbf{v} + \mathbf{w}) &= (\mathbf{v}_1 + \mathbf{w}_1)(\mathbf{v}_1 + \mathbf{w}_1) + (\mathbf{v}_2 + \mathbf{w}_2)(\mathbf{v}_2 + \mathbf{w}_2) = \mathbf{v}_1^2 + 2\mathbf{v}_1\mathbf{w}_1 + \mathbf{w}_1^2 + \mathbf{v}_2^2 + 2\mathbf{v}_2\mathbf{w}_2 + \mathbf{w}_2^2\\ &= (\mathbf{v}_1^2 + \mathbf{v}_2^2) + (2\mathbf{v}_1\mathbf{w}_1 + 2\mathbf{v}_2\mathbf{w}_2) + (\mathbf{w}_1^2 + \mathbf{w}_2^2) = \|\mathbf{v}\|^2 + 2(\mathbf{v} \bullet \mathbf{w}) + \|\mathbf{w}\|^2\end{aligned}$$

83. Using the result from exercise 82, we have $\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + 2(\mathbf{v} \bullet \mathbf{w}) + \|\mathbf{w}\|^2$. From exercise 80 (the Cauchy-Schwarz inequality), this becomes $\|\mathbf{v} + \mathbf{w}\|^2 \le \|\mathbf{v}\|^2 + 2\|\mathbf{v}\|\|\mathbf{w}\| + \|\mathbf{w}\|^2$. Then

$\|\mathbf{v} + \mathbf{w}\|^2 \le (\|\mathbf{v}\| + \|\mathbf{w}\|)^2 \Rightarrow \|\mathbf{v} + \mathbf{w}\| \le \|\mathbf{v}\| + \|\mathbf{w}\|$.

84. Given $\mathbf{v} = \langle \mathbf{v}_1, \mathbf{v}_2 \rangle$ and $\mathbf{w} = \langle \mathbf{w}_1, \mathbf{w}_2 \rangle$. First compute $\mathbf{v} \bullet \mathbf{w}$ and $\|\mathbf{w}\|^2$.

$\mathbf{v} \bullet \mathbf{w} = \mathbf{v}_1\mathbf{w}_1 + \mathbf{v}_2\mathbf{w}_2;\quad \|\mathbf{w}\|^2 = \mathbf{w}_1^2 + \mathbf{w}_2^2$

$$\mathbf{v_1} = \text{proj}_{\mathbf{w}}\ \mathbf{v} = \left(\frac{\mathbf{v} \bullet \mathbf{w}}{\|\mathbf{w}\|^2}\right)\mathbf{w} = \frac{\mathbf{v}_1\mathbf{w}_1 + \mathbf{v}_2\mathbf{w}_2}{\mathbf{w}_1^2 + \mathbf{w}_2^2}\langle \mathbf{w}_1, \mathbf{w}_2 \rangle = \left\langle \frac{\mathbf{v}_1\mathbf{w}_1^2 + \mathbf{v}_2\mathbf{w}_1\mathbf{w}_2}{\mathbf{w}_1^2 + \mathbf{w}_2^2}, \frac{\mathbf{v}_1\mathbf{w}_1\mathbf{w}_2 + \mathbf{v}_2\mathbf{w}_2^2}{\mathbf{w}_1^2 + \mathbf{w}_2^2} \right\rangle$$

(*continued on next page*)

(*continued*)

$$\mathbf{v_2} = \mathbf{v} - \mathbf{v_1} = \langle \mathbf{v}_1, \mathbf{v}_2 \rangle - \left\langle \frac{\mathbf{v}_1\mathbf{w}_1{}^2 + \mathbf{v}_2\mathbf{w}_1\mathbf{w}_2}{\mathbf{w}_1{}^2 + \mathbf{w}_2{}^2}, \frac{\mathbf{v}_1\mathbf{w}_1\mathbf{w}_2 + \mathbf{v}_2\mathbf{w}_2{}^2}{\mathbf{w}_1{}^2 + \mathbf{w}_2{}^2} \right\rangle$$
$$= \left\langle \frac{\mathbf{v}_1\left(\mathbf{w}_1{}^2 + \mathbf{w}_2{}^2\right) - \left(\mathbf{v}_1\mathbf{w}_1{}^2 + \mathbf{v}_2\mathbf{w}_1\mathbf{w}_2\right)}{\mathbf{w}_1{}^2 + \mathbf{w}_2{}^2}, \frac{\mathbf{v}_2\left(\mathbf{w}_1{}^2 + \mathbf{w}_2{}^2\right) - \left(\mathbf{v}_1\mathbf{w}_1\mathbf{w}_2 + \mathbf{v}_2\mathbf{w}_2{}^2\right)}{\mathbf{w}_1{}^2 + \mathbf{w}_2{}^2} \right\rangle$$
$$= \left\langle \frac{\mathbf{v}_1\mathbf{w}_1{}^2 + \mathbf{v}_1\mathbf{w}_2{}^2 - \mathbf{v}_1\mathbf{w}_1{}^2 - \mathbf{v}_2\mathbf{w}_1\mathbf{w}_2}{\mathbf{w}_1{}^2 + \mathbf{w}_2{}^2}, \frac{\mathbf{v}_2\mathbf{w}_1{}^2 + \mathbf{v}_2\mathbf{w}_2{}^2 - \mathbf{v}_1\mathbf{w}_1\mathbf{w}_2 - \mathbf{v}_2\mathbf{w}_2{}^2}{\mathbf{w}_1{}^2 + \mathbf{w}_2{}^2} \right\rangle$$
$$= \left\langle \frac{\mathbf{v}_1\mathbf{w}_2{}^2 - \mathbf{v}_2\mathbf{w}_1\mathbf{w}_2}{\mathbf{w}_1{}^2 + \mathbf{w}_2{}^2}, \frac{\mathbf{v}_2\mathbf{w}_1{}^2 - \mathbf{v}_1\mathbf{w}_1\mathbf{w}_2}{\mathbf{w}_1{}^2 + \mathbf{w}_2{}^2} \right\rangle$$

To show that $\mathbf{v_2}$ is orthogonal to **w**, show that the dot product equals 0.

$$\mathbf{v_2} \bullet \mathbf{w} = \left\langle \frac{\mathbf{v}_1\mathbf{w}_2{}^2 - \mathbf{v}_2\mathbf{w}_1\mathbf{w}_2}{\mathbf{w}_1{}^2 + \mathbf{w}_2{}^2}, \frac{\mathbf{v}_2\mathbf{w}_1{}^2 - \mathbf{v}_1\mathbf{w}_1\mathbf{w}_2}{\mathbf{w}_1{}^2 + \mathbf{w}_2{}^2} \right\rangle \bullet \langle \mathbf{w}_1, \mathbf{w}_2 \rangle = \left(\frac{\mathbf{v}_1\mathbf{w}_2{}^2 - \mathbf{v}_2\mathbf{w}_1\mathbf{w}_2}{\mathbf{w}_1{}^2 + \mathbf{w}_2{}^2} \right)\mathbf{w}_1 + \left(\frac{\mathbf{v}_2\mathbf{w}_1{}^2 - \mathbf{v}_1\mathbf{w}_1\mathbf{w}_2}{\mathbf{w}_1{}^2 + \mathbf{w}_2{}^2} \right)\mathbf{w}_2$$
$$= \frac{\mathbf{v}_1\mathbf{w}_1\mathbf{w}_2{}^2 - \mathbf{v}_2\mathbf{w}_1{}^2\mathbf{w}_2}{\mathbf{w}_1{}^2 + \mathbf{w}_2{}^2} + \frac{\mathbf{v}_2\mathbf{w}_1{}^2\mathbf{w}_2 - \mathbf{v}_1\mathbf{w}_1\mathbf{w}_2{}^2}{\mathbf{w}_1{}^2 + \mathbf{w}_2{}^2} = 0$$

85. $\cos\theta = \dfrac{\langle 2,-5\rangle \bullet \langle 5,2\rangle}{\|\langle 2,-5\rangle\|\|\langle 5,2\rangle\|} = \dfrac{0}{\|\langle 2,-5\rangle\|\|\langle 5,2\rangle\|} = 0 \Rightarrow$
$\cos^{-1} 0 = 90°$, so the vectors are perpendicular.

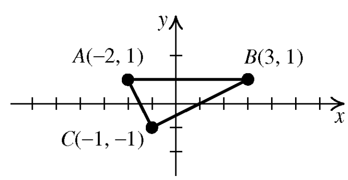
86. $\cos\theta = 0 = \dfrac{\langle 3,7\rangle \bullet \langle x,y\rangle}{\|\langle 3,7\rangle\|\|\langle x,y\rangle\|} = \dfrac{3x+7y}{\|\langle 3,7\rangle\|\|\langle x,y\rangle\|} \Rightarrow$
$3x + 7y = 0 \Rightarrow 3x = -7y$. If $x = 7$, then $y = -3$. So, $7\mathbf{i} - 3\mathbf{j}$ is a vector perpendicular to $3\mathbf{i} + 7\mathbf{j}$.

87. A vector orthogonal to $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$ is $\mathbf{w} = 4\mathbf{i} + 3\mathbf{j}$. (Check by showing that the dot product equals zero.) The unit vectors are
$\dfrac{1}{\|\mathbf{w}\|}\mathbf{w} = \dfrac{1}{\sqrt{4^2+3^2}}\langle 4,3\rangle = \pm\dfrac{1}{5}\langle 4,3\rangle = \pm\left\langle \dfrac{4}{5}, \dfrac{3}{5}\right\rangle.$

88. a. $\mathbf{v} \bullet \mathbf{u_1} = (x\mathbf{u_1} + y\mathbf{u_2}) \bullet \mathbf{u_1} = x\mathbf{u_1} \bullet \mathbf{u_1} + y\mathbf{u_2} \bullet \mathbf{u_1}$
Since $\mathbf{u_1}$ and $\mathbf{u_2}$ are orthogonal,
$\mathbf{u_2} \bullet \mathbf{u_1} = 0$, so $x\mathbf{u_1} \bullet \mathbf{u_1} + y\mathbf{u_2} \bullet \mathbf{u_1} = x\mathbf{u_1} \bullet \mathbf{u_1}$
Since $\mathbf{u_1}$ is a unit vector, $\mathbf{u_1} \bullet \mathbf{u_1} = 1$. Thus
$x\mathbf{u_1} \bullet \mathbf{u_1} = x$, and therefore $\mathbf{v} \bullet \mathbf{u_1} = x$.

b. $\mathbf{v} \bullet \mathbf{u}_2 = (x\mathbf{u_1} + y\mathbf{u_2}) \bullet \mathbf{u_2} = x\mathbf{u_1} \bullet \mathbf{u_2} + y\mathbf{u_2} \bullet \mathbf{u_2}$
Since $\mathbf{u_1}$ and $\mathbf{u_2}$ are orthogonal, $\mathbf{u_2} \bullet \mathbf{u_1} = 0$,
so $x\mathbf{u_1} \bullet \mathbf{u_2} + y\mathbf{u_2} \bullet \mathbf{u_2} = y\mathbf{u_2} \bullet \mathbf{u_2}$
Since $\mathbf{u_2}$ is a unit vector, $\mathbf{u_2} \bullet \mathbf{u_2} = 1$. Thus
$y\mathbf{u_2} \bullet \mathbf{u_2} = y$, and therefore $\mathbf{v} \bullet \mathbf{u_2} = y$.

89. The vectors that make up the sides of the triangles are $\overrightarrow{AB} = \langle 5,0\rangle$, $\overrightarrow{AC} = \langle 1,-2\rangle$, and $\overrightarrow{BC} = \langle -4,-2\rangle$.



Use the dot product to identify if two vectors are orthogonal.
$\overrightarrow{AC} \bullet \overrightarrow{BC} = (1)(-4) + (-2)(-2) = 0$
Since $\overrightarrow{AC} \bullet \overrightarrow{BC} = 0, \overrightarrow{AC} \perp \overrightarrow{BC}$. Thus $\triangle ABC$ is a right triangle.

90.
$$\begin{aligned}
&\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 \\
&= (\mathbf{u}+\mathbf{v}) \bullet (\mathbf{u}+\mathbf{v}) + (\mathbf{u}-\mathbf{v}) \bullet (\mathbf{u}-\mathbf{v}) \\
&= \mathbf{u} \bullet \mathbf{u} + \mathbf{u} \bullet \mathbf{v} + \mathbf{v} \bullet \mathbf{u} + \mathbf{v} \bullet \mathbf{v} \\
&\qquad + \mathbf{u} \bullet \mathbf{u} - \mathbf{u} \bullet \mathbf{v} - \mathbf{v} \bullet \mathbf{u} + \mathbf{v} \bullet \mathbf{v} \\
&= \|\mathbf{u}\|^2 + 2\mathbf{u} \bullet \mathbf{v} + \|\mathbf{v}\|^2 + \|\mathbf{u}\|^2 - 2\mathbf{u} \bullet \mathbf{v} + \|\mathbf{v}\|^2 \\
&= 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2
\end{aligned}$$

91.
$$\begin{aligned}
&\|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2 \\
&= (\mathbf{u}+\mathbf{v}) \bullet (\mathbf{u}+\mathbf{v}) - (\mathbf{u}-\mathbf{v}) \bullet (\mathbf{u}-\mathbf{v}) \\
&= \mathbf{u} \bullet \mathbf{u} + \mathbf{u} \bullet \mathbf{v} + \mathbf{v} \bullet \mathbf{u} + \mathbf{v} \bullet \mathbf{v} \\
&\qquad - (\mathbf{u} \bullet \mathbf{u} - \mathbf{u} \bullet \mathbf{v} - \mathbf{v} \bullet \mathbf{u} + \mathbf{v} \bullet \mathbf{v}) \\
&= \|\mathbf{u}\|^2 + 2(\mathbf{u} \bullet \mathbf{v}) + \|\mathbf{v}\|^2 - \|\mathbf{u}\|^2 + 2(\mathbf{u} \bullet \mathbf{v}) - \|\mathbf{v}\|^2 \\
&= 4(\mathbf{u} \bullet \mathbf{v})
\end{aligned}$$

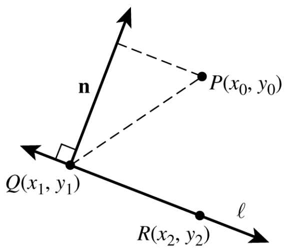
92. If $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}-\mathbf{v}\|$, then

$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}-\mathbf{v}\|^2 \Rightarrow$

$\|\mathbf{u}\|^2 + 2\mathbf{u}\bullet\mathbf{v} + \|\mathbf{v}\|^2 = \|\mathbf{u}\|^2 - 2\mathbf{u}\bullet\mathbf{v} + \|\mathbf{v}\|^2 \Rightarrow$

$2\mathbf{u}\bullet\mathbf{v} = -2\mathbf{u}\bullet\mathbf{v}$, which is true if and only if $\mathbf{u}\bullet\mathbf{v} = 0$. Thus **u** and **v** must be orthogonal.

93.



$\overrightarrow{QR} = \langle x_2 - x_1,\ y_2 - y_1\rangle$

$ax + by + c = 0 \Rightarrow y = -\frac{a}{b}x - \frac{c}{b}$

$m = -\frac{a}{b} = \frac{y_2 - y_1}{x_2 - x_1} \Rightarrow a(x_2 - x_1) = -b(y_2 - y_1)$

$$\begin{aligned}\mathbf{n}\bullet\overrightarrow{QR} &= \langle x_2 - x_1,\ y_2 - y_1\rangle \bullet \langle a,\ b\rangle \\ &= a(x_2 - x_1) + b(y_2 - y_1) \\ &= -b(y_2 - y_1) + b(y_2 - y_1) = 0\end{aligned}$$

Thus, $\mathbf{n} \perp \overrightarrow{QR}$.

$\mathbf{n} = \langle a,\ b\rangle = \langle x_0 - x,\ y_0 - y\rangle$

$$\begin{aligned}\left\|\text{proj}_{\mathbf{n}}\ \overrightarrow{QP}\right\| &= \frac{\overrightarrow{QP}\bullet\mathbf{n}}{\|\mathbf{n}\|} = \frac{a(x_0 - x_1) + b(y_0 - y_1)}{\sqrt{a^2 + b^2}} \\ &= \frac{ax_0 + by_0 - (ax_1 + by_1)}{\sqrt{a^2 + b^2}}\end{aligned}$$

Since $Q(x_1,\ y_1)$ is on ℓ, $ax_1 + by_1 = -c$.

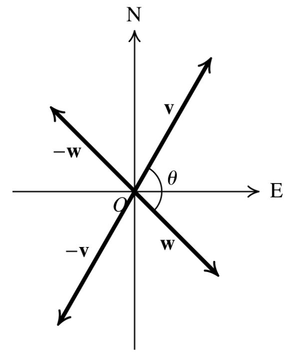
$$\text{Thus, } \left\|\text{proj}_{\mathbf{n}}\ \overrightarrow{QP}\right\| = \frac{ax_0 + by_0 - (ax_1 + by_1)}{\sqrt{a^2 + b^2}} = \frac{ax_0 + by_0 + c}{\sqrt{a^2 + b^2}} = d$$

7.5 Critical Thinking/Discussion/Writing

94. False. Let $\mathbf{u} = \langle 1, 0\rangle$, $\mathbf{v} = \langle 0, 2\rangle$, and $\mathbf{w} = \langle 0, 3\rangle$. Then $\mathbf{u}\bullet\mathbf{v} = 0$ and $\mathbf{u}\bullet\mathbf{w} = 0$. However, $\mathbf{v} \neq \mathbf{w}$.

95. Since $\cos\theta = \frac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$, the quotient must be a number between −1 and 1. (See page 686 in the text for the proof.)

96. Yes. If we graph **v** and **w** along with **−v** and **−w** on the same coordinate plane, we see that the angles formed are vertical angles. Thus, the angles are equal.



7.5 Maintaining Skills

97. $x = -1$ is the graph of a vertical line with x-intercept −1 and no y-intercept.

98. $y = 2$ is the graph of a horizontal line with no x-intercept and y-intercept 2.

99. $y = x^2 - x - 6$

$0 = x^2 - x - 6 \Rightarrow 0 = (x+2)(x-3) \Rightarrow$

$x = -2,\ 3$

$y = 0^2 - 0 - 6 = -6$

$y = x^2 - x - 6$ is the graph of a parabola with x-intercepts −2 and 3 and y-intercept −6

100. $x = y^2 + 4y + 3$

$0 = y^2 + 4y + 3 \Rightarrow 0 = (y+3)(y+1) \Rightarrow$

$y = -3,\ -1$

$x = 0^2 + 4(0) + 3 = 3$

$x = y^2 + 4y + 3$ is the graph of a parabola with x-intercept 3 and y-intercepts −3 and −1.

101. $x^2 + (y-1)^2 = 4$

$x^2 + (0-1)^2 = 4 \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$

$0^2 + (y-1)^2 = 4 \Rightarrow (y-1)^2 = 4 \Rightarrow$

$y - 1 = \pm 2 \Rightarrow y = 1 \pm 2 = -1,\ 3$

$x^2 + (y-1)^2 = 4$ is the graph of a circle with center (0, 1) and radius 2. The x-intercepts are $\pm\sqrt{3}$ and the y-intercepts are −1 and 3.

102. $(x+1)^2 + y^2 = 9$
$(x+1)^2 + 0^2 = 9 \Rightarrow (x+1)^2 = 9 \Rightarrow$
$x+1 = \pm 3 \Rightarrow x = -1 \pm 3 = -4,\ 2$
$(0+1)^2 + y^2 = 9 \Rightarrow y^2 = 8 \Rightarrow y = \pm 2\sqrt{2}$
$(x+1)^2 + y^2 = 9$ is the graph of a circle with center (−1, 0) and radius 3. The *x*-intercepts are −4 and 2 and the *y*-intercepts are $\pm 2\sqrt{2}$.

103. $x^2 + y^2 - 2x + 4y + 2 = 0 \Rightarrow$
$x^2 - 2x + y^2 + 4y = -2 \Rightarrow$
$x^2 - 2x + 1 + y^2 + 4y + 4 = -2 + 1 + 4 \Rightarrow$
$(x-1)^2 + (y+2)^2 = 3$
$x^2 + 0^2 - 2x + 4(0) + 2 = 0 \Rightarrow$
$x^2 - 2x + 2 = 0 \Rightarrow x^2 - 2x + 1 = -2 + 1 \Rightarrow$
$(x-1)^2 = -1,$ which has no real solution.
$0^2 + y^2 - 2(0) + 4y + 2 = 0 \Rightarrow$
$y^2 + 4y + 4 = -2 + 4 \Rightarrow (y+2)^2 = 2 \Rightarrow$
$y + 2 = \pm\sqrt{2} \Rightarrow y = -2 \pm \sqrt{2}$
$x^2 + y^2 - 2x + 4y + 2 = 0$ is the graph of a circle with center (1, −2) and radius $\sqrt{3}$. There are no *x*-intercepts and the *y*-intercepts are $-2 \pm \sqrt{2}$.

104. $x^2 + y^2 + 6x - 4y - 12 = 0 \Rightarrow$
$x^2 + 6x + y^2 - 4y = 12 \Rightarrow$
$x^2 + 6x + 9 + y^2 - 4y + 4 = 12 + 9 + 4 \Rightarrow$
$(x+3)^2 + (y-2)^2 = 25$
$x^2 + 0^2 + 6x - 4(0) - 12 = 0 \Rightarrow$
$x^2 + 6x - 12 = 0 \Rightarrow x^2 + 6x + 9 = 12 + 9 \Rightarrow$
$(x+3)^2 = 21 \Rightarrow x + 3 = \pm\sqrt{21} \Rightarrow x = -3 \pm \sqrt{21}$
$0^2 + y^2 + 6(0) - 4y - 12 = 0 \Rightarrow$
$y^2 - 4y - 12 = 0 \Rightarrow (y+2)(y-6) = 0 \Rightarrow$
$y = -2,\ 6$
$x^2 + y^2 + 6x - 4y - 12 = 0$ is the graph of a circle with center (−3, 2) and radius 5. The *x*-intercepts are $-3 \pm \sqrt{21}$ and the *y*-intercepts are −2 and 6.

See section 2.2 in the text to review the tests for symmetry.

105. $x^3 + y^2 = 5$
Test for all three symmetries:
x-axis: $x^3 + (-y)^2 = 5 \Rightarrow x^3 + y^2 = 5,$ which is the same as the original equation, so the equation is symmetric with respect to the *x*-axis.
y-axis: $(-x)^3 + y^2 = 5 \Rightarrow -x^3 + y^2 = 5,$ which is not the same as the original equation. So the graph is not symmetric with respect to the *y*-axis.
origin: $(-x)^3 + (-y)^2 = 5 \Rightarrow -x^3 + y^2 = 5,$ which is not the same as the original equation. So the graph is not symmetric with respect to the origin.

106. $x^3 + y^3 - 3xy^2 = 0$
Test for all three symmetries:
x-axis: $x^3 + (-y)^3 - 3x(-y)^2 = 0 \Rightarrow$
$x^3 - y^3 - 3xy^2 = 0,$ which is not the same as the original equation, so the equation is not symmetric with respect to the *x*-axis.
y-axis: $(-x)^3 + y^3 - 3(-x)y^2 = 0 \Rightarrow$
$-x^3 + y^3 + 3xy^2 = 0,$ which is not the same as the original equation. So the graph is not symmetric with respect to the *y*-axis.
origin: $(-x)^3 + (-y)^3 - 3(-x)(-y)^2 = 0 \Rightarrow$
$-x^3 - y^3 + 3xy^2 = 0 \Rightarrow$
$(-1)\left(x^3 + y^3 - 3xy^2\right) = 0 \Rightarrow$
$x^3 + y^3 - 3xy^2 = 0,$ which is the same as the original equation. So the graph is symmetric with respect to the origin.

107. $2x^2 - 3y^3 = 7$
Test for all three symmetries:
x-axis: $2x^2 - 3(-y)^3 = 7 \Rightarrow 2x^2 + 3y^3 = 7,$ which is not the same as the original equation, so the equation is not symmetric with respect to the *x*-axis.
y-axis: $2(-x)^2 - 3y^3 = 7 \Rightarrow 2x^2 - 3y^3 = 7,$ which is the same as the original equation. So the graph is symmetric with respect to the *y*-axis.

(*continued on next page*)

(*continued*)

origin:

$2(-x)^2 - 3(-y)^3 = 7 \Rightarrow 2x^2 + 3y^3 = 7,$

which is not the same as the original equation. So the graph is not symmetric with respect to the origin.

108. $2x^2 + 3y^2 = 6$

Test for all three symmetries:

x-axis: $2x^2 + 3(-y)^2 = 6 \Rightarrow 2x^2 + 3y^2 = 6,$

which is the same as the original equation, so the equation is symmetric with respect to the x-axis.

y-axis: $2(-x)^2 + 3y^2 = 6 \Rightarrow 2x^2 + 3y^2 = 6,$

which is the same as the original equation. So the graph is symmetric with respect to the y-axis.

origin:

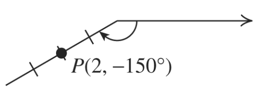
$2(-x)^2 + 3(-y)^2 = 6 \Rightarrow 2x^2 + 3y^2 = 6,$

which is the same as the original equation. So the graph is symmetric with respect to the origin.

7.6 Polar Coordinates

7.6 Practice Problems

1. a



b. (−2, 30º) **c.** (2, 210º)

d. (−2, −330º)

2. a. $(-3, 60°) \Rightarrow x = -3\cos 60° = -\frac{3}{2}$

$y = -3\sin 60° = -\frac{3\sqrt{3}}{2}$

The rectangular coordinates of $(-3, 60°)$ are $\left(-\frac{3}{2}, -\frac{3\sqrt{3}}{2}\right)$.

b. $\left(2, -\frac{\pi}{4}\right) \Rightarrow x = 2\cos\left(-\frac{\pi}{4}\right) = \sqrt{2}$

$y = 2\sin\left(-\frac{\pi}{4}\right) = -\sqrt{2}$

The rectangular coordinates of $\left(\pi, -\frac{\pi}{4}\right)$ are $\left(\sqrt{2}, -\sqrt{2}\right)$.

3. The point (−1, −1) lies in quadrant III with $x = -1$ and $y = -1$.

$(-1, -1) \Rightarrow r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

$\tan\theta = \frac{y}{x} = 1 \Rightarrow \theta = \frac{\pi}{4}$ or $\theta = \frac{5\pi}{4}$

$(-1, -1)$ is in Quadrant III, so choose $\theta = \frac{5\pi}{4}$.

The polar coordinates of $(-1, -1)$ are $\left(\sqrt{2}, \frac{5\pi}{4}\right)$.

4. $OE = 15$ in., $EH = 10$ in.

$\overrightarrow{OE} = \langle 15\cos 30°, 15\sin 30° \rangle$

$\overrightarrow{EH} = \langle 10\cos 45°, 10\sin 45° \rangle$

$\overrightarrow{OH} = \overrightarrow{OE} + \overrightarrow{EH}$

$= \langle 15\cos 30° + 10\cos 45°, 15\sin 30° + 10\sin 45° \rangle$

$$r = \sqrt{(15\cos 30° + 10\cos 45°)^2 + (15\sin 30° + 10\sin 45°)^2}$$

≈ 24.8 in.

$\theta = \tan^{-1}\left(\frac{15\cos 30° + 10\cos 45°}{15\sin 30° + 10\sin 45°}\right) \approx 36.0°$

The hand is at $(24.8, 36.0°)$ relative to the shoulder.

5.

$$\begin{aligned} x^2 + y^2 - 2y + 3 &= 0 \\ (r\cos\theta)^2 + (r\sin\theta)^2 - 2r\sin\theta + 3 &= 0 \\ r^2(\cos^2\theta + \sin^2\theta) - 2r\sin\theta + 3 &= 0 \\ r^2 - 2r\sin\theta + 3 &= 0 \end{aligned}$$

6. a. $r = -5 \Rightarrow r^2 = 25 \Rightarrow x^2 + y^2 = 25$.

A circle with center (0, 0) and radius 5.

b. $\theta = -\frac{\pi}{3} \Rightarrow \tan\theta = -\sqrt{3} = \frac{y}{x} \Rightarrow y = -\sqrt{3}x$.

A line through the origin with slope $-\sqrt{3}$.

c. $r = \sec\theta = \frac{1}{\cos\theta} \Rightarrow r\cos\theta = 1 \Rightarrow x = 1$

A vertical line through $x = 1$.

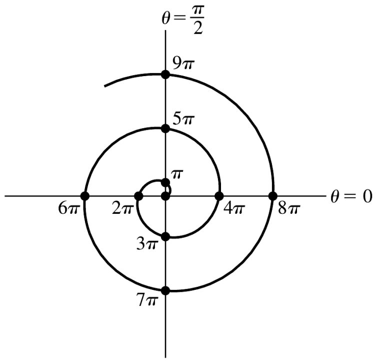
d. $r = 2\sin\theta \Rightarrow r^2 = 2r\sin\theta \Rightarrow$

$x^2 + y^2 = 2y \Rightarrow x^2 + y^2 - 2y = 0 \Rightarrow$

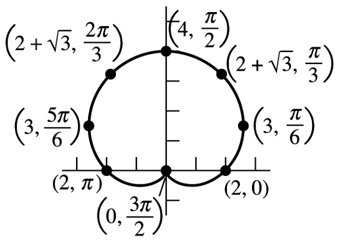
$x^2 + y^2 - 2y + 1 = 1 \Rightarrow x^2 + (y-1)^2 = 1$

A circle with center (0, 1) and radius 1.

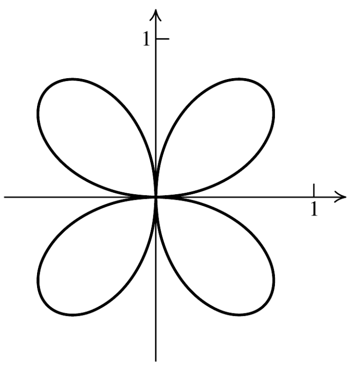
7.



8.



9.



7.6 Basic Concepts and Skills

1. The polar coordinates (r, θ) of a point are the directed distance r to the <u>origin</u> and the directed angle θ from the polar <u>axis</u>.

2. The positive value of r corresponds to a distance r on the terminal sides of θ, and a negative value corresponds to a distance $|r|$ in the <u>opposite</u> direction.

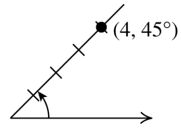
3. The substitutions $x = \underline{r\cos\theta}$ and $y = \underline{r\sin\theta}$ transform a rectangular equation into a polar equation for the same curve.

4. Polar coordinates are found from the rectangular coordinates by the formulas $r = \underline{\sqrt{x^2 + y^2}}$ and $\underline{\theta = \tan^{-1}\left(\frac{y}{x}\right) \text{ in the correct quadrant.}}$

5. True

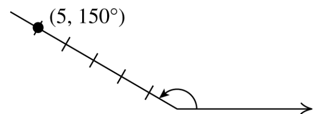
6. True

7.



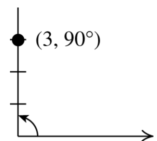
a. $(-4, 225°)$ **b.** $(-4, -135°)$
c. $(4, -315°)$

8.



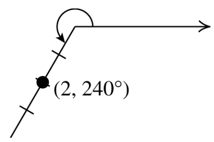
a. $(-5, 330°)$ **b.** $(-5, -30°)$
c. $(5, -210°)$

9.



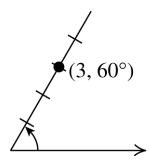
a. $(-3, 270°)$ **b.** $(-3, -90°)$
c. $(3, -270°)$

10.



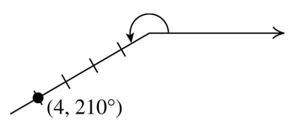
a. $(-2, 60°)$ **b.** $(-2, -300°)$
c. $(2, -120°)$

11.

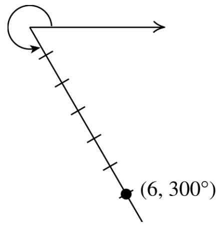


a. $(-3, 240°)$ **b.** $(-3, -120°)$
c. $(3, -300°)$

12.

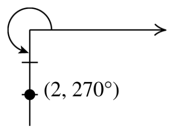


a. $(-4, 30°)$ **b.** $(-4, -330°)$
c. $(4, -150°)$

13. 

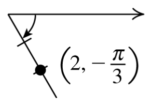
a. $(-6, 120°)$ **b.** $(-6, -240°)$

c. $(6, -60°)$

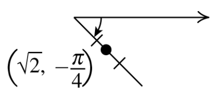
14. 

a. $(-2, 90°)$ **b.** $(-2, -270°)$

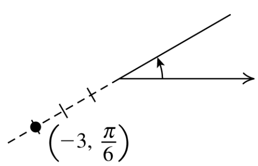
c. $(2, -90°)$

15. 

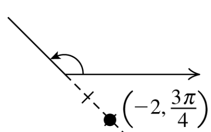
a. $\left(2, \frac{5\pi}{3}\right)$ **b.** $\left(-2, \frac{2\pi}{3}\right)$

16. 

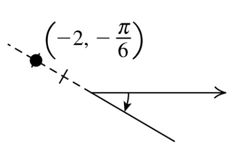
a. $\left(\sqrt{2}, \frac{7\pi}{4}\right)$ **b.** $\left(-\sqrt{2}, \frac{3\pi}{4}\right)$

17. 

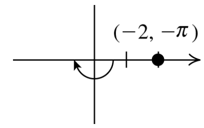
a. $\left(-3, -\frac{11\pi}{6}\right)$ **b.** $\left(3, -\frac{5\pi}{6}\right)$

18. 

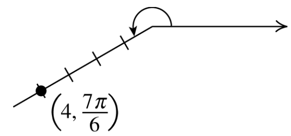
a. $\left(-2, -\frac{5\pi}{4}\right)$ **b.** $\left(2, -\frac{\pi}{4}\right)$

19. 

a. $\left(-2, \frac{11\pi}{6}\right)$ **b.** $\left(2, \frac{5\pi}{6}\right)$

20. 

a. $(-2, \pi)$ **b.** $(2, 0)$

21. 

a. $\left(4, -\frac{5\pi}{6}\right)$ **b.** $\left(-4, \frac{\pi}{6}\right)$

22. 

a. $(2, -3.28)$ **b.** $(-2, -0.14)$

23. $(3, 60°) \Rightarrow x = 3\cos 60° = \frac{3}{2}, y = 3\sin 60° = \frac{3\sqrt{3}}{2}$

The rectangular coordinates of $(3, 60°)$ are $\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$.

24. $(-2, -30°) \Rightarrow x = -2\cos(-30°) = -\sqrt{3},$
$y = -2\sin(-30°) = 1$

The rectangular coordinates of $(-2, -30°)$ are $\left(-\sqrt{3}, 1\right)$.

25. $(5, -60°) \Rightarrow x = 5\cos(-60°) = \frac{5}{2},$
$y = 5\sin(-60°) = -\frac{5\sqrt{3}}{2}$

The rectangular coordinates of $(5, -60°)$ are $\left(\frac{5}{2}, -\frac{5\sqrt{3}}{2}\right)$.

26. $(-3, 90°) \Rightarrow x = -3\cos 90° = 0,$
$y = -3\sin 90° = -3$

The rectangular coordinates of $(-3, 90°)$ are $(0, -3)$.

27. $(3, \pi) \Rightarrow x = 3\cos\pi = -3, y = 3\sin\pi = 0$

The rectangular coordinates of $(3, \pi)$ are $(-3, 0)$.

28. $\left(\sqrt{2},-\frac{\pi}{4}\right) \Rightarrow x=\sqrt{2}\cos\left(-\frac{\pi}{4}\right)=1,$
$y=\sqrt{2}\sin\left(-\frac{\pi}{4}\right)=-1$

The rectangular coordinates of $\left(\sqrt{2},-\frac{\pi}{4}\right)$ are $(1,-1)$.

29. $\left(-2,-\frac{5\pi}{6}\right) \Rightarrow x=-2\cos\left(-\frac{5\pi}{6}\right)=\sqrt{3},$
$y=-2\sin\left(-\frac{5\pi}{6}\right)=1$

The rectangular coordinates of $\left(-2,-\frac{5\pi}{6}\right)$ are $\left(\sqrt{3},1\right)$.

30. $\left(-1,\frac{7\pi}{6}\right) \Rightarrow x=-\cos\left(\frac{7\pi}{6}\right)=\frac{\sqrt{3}}{2},$
$y=-\sin\left(\frac{7\pi}{6}\right)=\frac{1}{2}$

The rectangular coordinates of $\left(-1,\frac{7\pi}{6}\right)$ are $\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$.

31. $(1,-1) \Rightarrow r=\sqrt{1^2+(-1)^2}=\sqrt{2}$
$\tan\theta=\frac{y}{x}=-1 \Rightarrow \theta=\frac{3\pi}{4}$ or $\theta=\frac{7\pi}{4}$
$(1,-1)$ is in Quadrant IV, so choose $\theta=\frac{7\pi}{4}$.
The polar coordinates of $(1,-1)$ are $\left(\sqrt{2},\frac{7\pi}{4}\right)$.

32. $\left(-\sqrt{3},1\right) \Rightarrow r=\sqrt{\left(-\sqrt{3}\right)^2+1^2}=2$
$\tan\theta=\frac{y}{x}=-\frac{\sqrt{3}}{3} \Rightarrow \theta=\frac{5\pi}{6}$ or $\theta=\frac{11\pi}{6}$
$\left(-\sqrt{3},1\right)$ is in Quadrant II, so choose $\theta=5\pi/6$.
The polar coordinates of $\left(-\sqrt{3},1\right)$ are $\left(2,\frac{5\pi}{6}\right)$.

33. $(3,3) \Rightarrow r=\sqrt{3^2+3^2}=3\sqrt{2}$
$\tan\theta=\frac{y}{x}=1 \Rightarrow \theta=\frac{\pi}{4}$ or $\theta=\frac{5\pi}{4}$
$(3,3)$ is in Quadrant I, so choose $\theta=\frac{\pi}{4}$.
The polar coordinates of $(3,3)$ are $\left(3\sqrt{2},\frac{\pi}{4}\right)$.

34. $(-4,0) \Rightarrow r=\sqrt{(-4)^2+0^2}=4$
$\tan\theta=\frac{y}{x}=0 \Rightarrow \theta=0$ or $\theta=\pi$
$(-4,0)$ is on the negative x-axis, so choose $\theta=\pi$.
The polar coordinates of $(-4,0)$ are $(4,\pi)$.

35. $(3,-3) \Rightarrow r=\sqrt{3^2+(-3)^2}=3\sqrt{2}$
$\tan\theta=\frac{y}{x}=-1 \Rightarrow \theta=\frac{3\pi}{4}$ or $\theta=\frac{7\pi}{4}$
$(3,-3)$ is in Quadrant IV, so choose $\theta=\frac{7\pi}{4}$.
The polar coordinates of $(3,-3)$ are $\left(3\sqrt{2},\frac{7\pi}{4}\right)$.

36. $\left(2\sqrt{3},2\right) \Rightarrow r=\sqrt{\left(2\sqrt{3}\right)^2+2^2}=4$
$\tan\theta=\frac{y}{x}=\frac{\sqrt{3}}{3} \Rightarrow \theta=\frac{\pi}{6}$ or $\theta=\frac{7\pi}{6}$
$\left(2\sqrt{3},2\right)$ is in Quadrant I, so choose $\theta=\frac{\pi}{6}$.
The polar coordinates of $\left(2\sqrt{3},2\right)$ are $\left(4,\frac{\pi}{6}\right)$.

37. $\left(-1,\sqrt{3}\right) \Rightarrow r=\sqrt{(-1)^2+\left(\sqrt{3}\right)^2}=2$
$\tan\theta=\frac{y}{x}=-\sqrt{3} \Rightarrow \theta=\frac{2\pi}{3}$ or $\theta=\frac{5\pi}{3}$
$\left(-1,\sqrt{3}\right)$ is in Quadrant II, so choose $\theta=\frac{2\pi}{3}$.
The polar coordinates of $\left(-1,\sqrt{3}\right)$ are $\left(2,\frac{2\pi}{3}\right)$.

38. $\left(-2,-2\sqrt{3}\right) \Rightarrow r=\sqrt{(-2)^2+\left(-2\sqrt{3}\right)^2}=4$
$\tan\theta=\frac{y}{x}=\sqrt{3} \Rightarrow \theta=\frac{\pi}{3}$ or $\theta=\frac{4\pi}{3}$
$\left(-2,-2\sqrt{3}\right)$ is in Quadrant III, so choose $\theta=\frac{4\pi}{3}$.
The polar coordinates of $\left(-2,-2\sqrt{3}\right)$ are $\left(4,\frac{4\pi}{3}\right)$.

39. $x^2 + y^2 = 16 \Rightarrow$
$(r\cos\theta)^2 + (r\sin\theta)^2 = 16 \Rightarrow$
$r^2(\cos^2\theta + \sin^2\theta) = 16 \Rightarrow r^2 = 16 \Rightarrow r = 4$

40. $x + y = 1 \Rightarrow r\cos\theta + r\sin\theta = 1 \Rightarrow$
$r(\cos\theta + \sin\theta) = 1 \Rightarrow r = \dfrac{1}{\cos\theta + \sin\theta}$

41. $y^2 = 4x \Rightarrow (r\sin\theta)^2 = 4(r\cos\theta) \Rightarrow$
$r^2\sin^2\theta = 4r\cos\theta \Rightarrow r = \dfrac{4\cos\theta}{\sin^2\theta} = 4\cot\theta\csc\theta$

42. $x^3 = 3y^2 \Rightarrow (r\cos\theta)^3 = 3(r\sin\theta)^2 \Rightarrow$
$r = \dfrac{3\sin^2\theta}{\cos^3\theta} = 3\tan^2\theta\sec\theta$

43. $y^2 = 6y - x^2 \Rightarrow x^2 + y^2 = 6y \Rightarrow$
$(r\cos\theta)^2 + (r\sin\theta)^2 = 6r\sin\theta \Rightarrow$
$r^2 = 6r\sin\theta \Rightarrow r = 6\sin\theta$

44. $x^2 - y^2 = 1 \Rightarrow$
$(r\cos\theta)^2 - (r\sin\theta)^2 = 1 \Rightarrow$
$r^2(\cos^2\theta - \sin^2\theta) = 1 \Rightarrow r^2 = \dfrac{1}{\cos^2\theta - \sin^2\theta}$

45. $xy = 1 \Rightarrow (r\cos\theta)(r\sin\theta) = 1 \Rightarrow$
$r^2 = \dfrac{1}{\sin\theta\cos\theta}$

46. $x^2 + y^2 - 4x + 6y = 12 \Rightarrow$
$(r\cos\theta)^2 + (r\sin\theta)^2 - 4r\cos\theta + 6r\sin\theta = 12 \Rightarrow$
$r^2(\cos^2\theta + \sin^2\theta) - 4r\cos\theta + 6r\sin\theta = 12 \Rightarrow$
$r^2 - 4r\cos\theta + 6r\sin\theta = 12$

47. $r = 2 \Rightarrow r^2 = 4 \Rightarrow x^2 + y^2 = 4$.
A circle with center (0, 0) and radius 2.

48. $r = -3 \Rightarrow r^2 = 9 \Rightarrow x^2 + y^2 = 9$.
A circle with center (0, 0) and radius 3.

49. $\theta = \dfrac{3\pi}{4} \Rightarrow \tan\theta = \tan\dfrac{3\pi}{4} = -1 \Rightarrow \dfrac{y}{x} = -1 \Rightarrow$
$y = -x$. A line with slope −1 and y-intercept 0.

50. $\theta = -\dfrac{\pi}{6} \Rightarrow \tan\theta = -\dfrac{\sqrt{3}}{3} = \dfrac{y}{x} \Rightarrow y = -\dfrac{\sqrt{3}}{3}x$.
A line with slope $-\dfrac{\sqrt{3}}{3}$ and y-intercept 0.

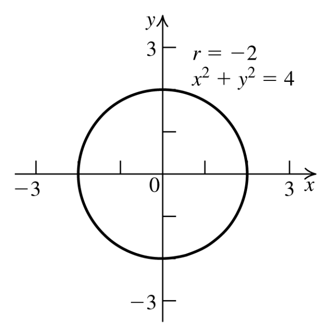
51. $r = 4\cos\theta \Rightarrow r^2 = 4r\cos\theta \Rightarrow x^2 + y^2 = 4x \Rightarrow$
$x^2 - 4x + y^2 = 0 \Rightarrow x^2 - 4x + 4 + y^2 = 4 \Rightarrow$
$(x-2)^2 + y^2 = 4$.
A circle with center (2, 0) and radius 2.

52. $r = 4\sin\theta \Rightarrow r^2 = 4r\sin\theta \Rightarrow x^2 + y^2 = 4y \Rightarrow$
$x^2 + y^2 - 4y = 0 \Rightarrow x^2 + y^2 - 4y + 4 = 4 \Rightarrow$
$x^2 + (y-2)^2 = 4$.
A circle with center (0, 2) and radius 2.

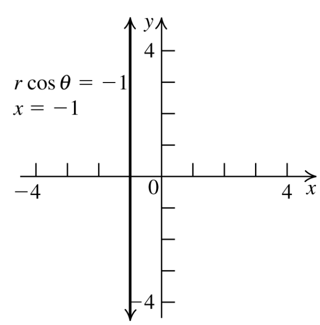
53. $r = -2\sin\theta \Rightarrow r^2 = -2r\sin\theta \Rightarrow$
$x^2 + y^2 = -2y \Rightarrow x^2 + y^2 + 2y = 0 \Rightarrow$
$x^2 + y^2 + 2y + 1 = 1 \Rightarrow$
$x^2 + (y+1)^2 = 1$.
A circle with center (0, −1) and radius 1.

54. $r = -3\cos\theta \Rightarrow r^2 = -3r\cos\theta \Rightarrow$
$x^2 + y^2 = -3x \Rightarrow x^2 + 3x + y^2 = 0 \Rightarrow$
$x^2 + 3x + \dfrac{9}{4} + y^2 = \dfrac{9}{4} \Rightarrow \left(x + \dfrac{3}{2}\right)^2 + y^2 = \dfrac{9}{4}$
A circle with center $\left(-\dfrac{3}{2}, 0\right)$ and radius $\dfrac{3}{2}$

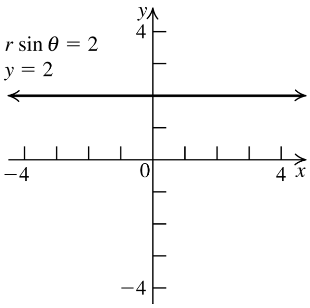
55. $r = -2 \Rightarrow r^2 = 4 \Rightarrow x^2 + y^2 = 4$



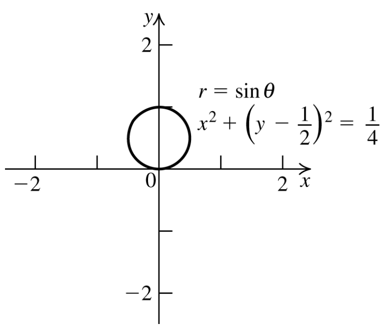
56. $r\cos\theta = -1 \Rightarrow x = -1$



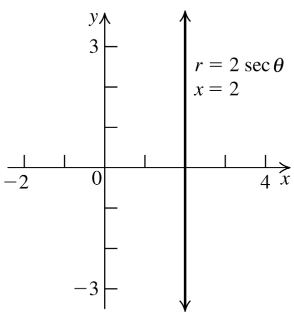
57. $r\sin\theta = 2 \Rightarrow y = 2$



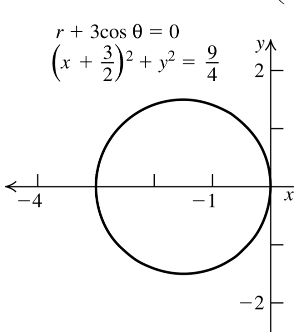
58. $r = \sin\theta \Rightarrow r^2 = r\sin\theta \Rightarrow x^2 + y^2 = y \Rightarrow$
$x^2 + y^2 - y + \frac{1}{4} = \frac{1}{4} \Rightarrow x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$



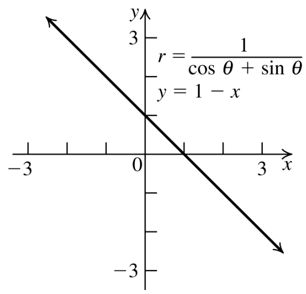
59. $r = 2\sec\theta = \dfrac{2}{\cos\theta} \Rightarrow r\cos\theta = 2 \Rightarrow x = 2$



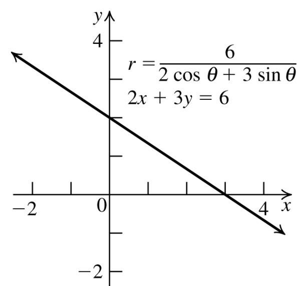
60. $r + 3\cos\theta = 0 \Rightarrow r = -3\cos\theta \Rightarrow$
$r^2 = -3r\cos\theta \Rightarrow x^2 + y^2 = -3x \Rightarrow$
$x^2 + 3x + \frac{9}{4} + y^2 = \frac{9}{4} \Rightarrow \left(x + \frac{3}{2}\right)^2 = \frac{9}{4}$



61. $r = \dfrac{1}{\cos\theta + \sin\theta} \Rightarrow r\cos\theta + r\sin\theta = 1 \Rightarrow$
$x + y = 1$

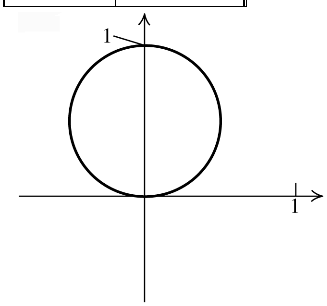


62. $r = \dfrac{6}{2\cos\theta + 3\sin\theta} \Rightarrow 2r\cos\theta + 3r\sin\theta = 6 \Rightarrow$
$2x + 3y = 6$



63.

θ	$\sin\theta$	θ	$\sin\theta$
0	0	π	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{5\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2} \approx 0.7071$	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2} \approx 0.7071$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2} \approx 0.8660$	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2} \approx 0.8660$
$\frac{\pi}{2}$	1		

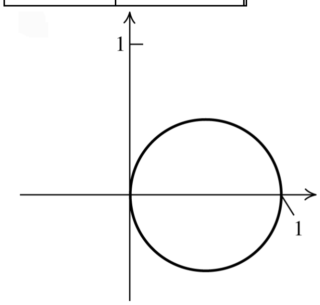


Circle

64.

θ	$\cos\theta$	θ	$\cos\theta$
0	1	π	-1
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}\approx 0.8660$	$\frac{5\pi}{6}$	$-\frac{\sqrt{3}}{2}\approx -0.8660$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}\approx 0.7071$	$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}\approx -0.7071$
$\pi/3$	$1/2$	$2\pi/3$	$-1/2$
$\pi/2$	0		

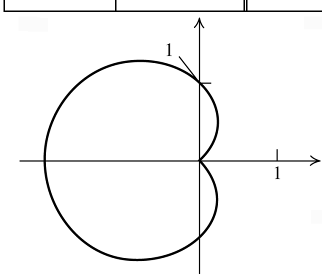
Circle



65.

θ	$1-\cos\theta$	θ	$1-\cos\theta$
0	0	π	2
$\frac{\pi}{6}$	$1-\frac{\sqrt{3}}{2}\approx 0.1340$	$-\frac{\pi}{6}$	$1-\frac{\sqrt{3}}{2}\approx 0.1340$
$\pi/3$	$1/2$	$-\pi/3$	$1/2$
$\pi/2$	1	$-\pi/2$	1
$\frac{2\pi}{3}$	$\frac{3}{2}$	$-\frac{2\pi}{3}$	$\frac{3}{2}$
$\frac{5\pi}{6}$	$1+\frac{\sqrt{3}}{2}\approx 1.8660$	$-\frac{5\pi}{6}$	$1+\frac{\sqrt{3}}{2}\approx 1.8660$

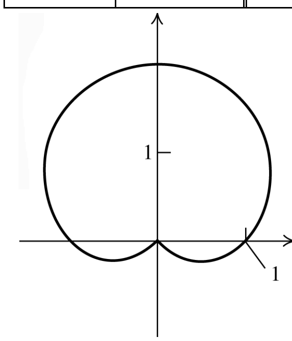
Limaçon



66.

θ	$1+\sin\theta$	θ	$1+\sin\theta$
0	1	π	1
$\frac{\pi}{6}$	$\frac{3}{2}$	$-\frac{\pi}{6}$	$\frac{1}{2}$
$\pi/3$	$1+\frac{\sqrt{3}}{2}\approx 1.8660$	$-\pi/3$	$1-\frac{\sqrt{3}}{2}\approx 0.1340$
$\pi/2$	2	$-\pi/2$	0
$\frac{2\pi}{3}$	$1+\frac{\sqrt{3}}{2}\approx 1.8660$	$-\frac{2\pi}{3}$	$1-\frac{\sqrt{3}}{2}\approx 0.1340$
$\frac{5\pi}{6}$	$\frac{3}{2}$	$-\frac{5\pi}{6}$	$\frac{1}{2}$

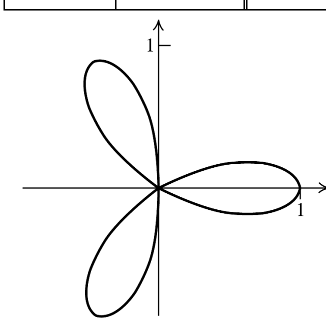
Limaçon



67.

θ	$\cos 3\theta$	θ	$\cos 3\theta$
0	1	π	1
$\frac{\pi}{6}$	0	$-\frac{\pi}{6}$	0
$\pi/3$	-1	$-\pi/3$	-1
$\pi/2$	0	$-\pi/2$	0
$\frac{2\pi}{3}$	1	$-\frac{2\pi}{3}$	1
$\frac{5\pi}{6}$	0	$-\frac{5\pi}{6}$	0

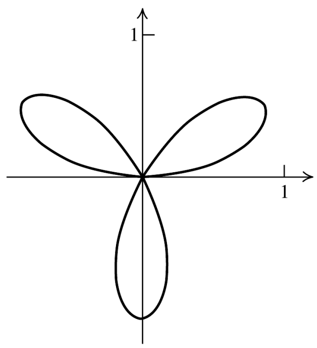
Rose curve



68.

θ	$\sin 3\theta$	θ	$\sin 3\theta$
0	0	π	0
$\frac{\pi}{6}$	1	$-\frac{\pi}{6}$	–1
$\pi/3$	0	$-\pi/3$	0
$\pi/2$	–1	$-\pi/2$	–
$\frac{2\pi}{3}$	0	$-\frac{2\pi}{3}$	0
$\frac{5\pi}{6}$	1	$-\frac{5\pi}{6}$	–1

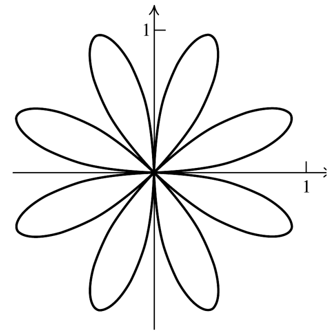
Rose curve



69.

θ	$\sin 4\theta$	θ	$\sin 4\theta$
0	0	π	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2} \approx 0.8660$	$-\frac{\pi}{6}$	$-\frac{\sqrt{3}}{2} \approx -0.8660$
$\pi/3$	$-\frac{\sqrt{3}}{2} \approx -0.8660$	$-\pi/3$	$\frac{\sqrt{3}}{2} \approx 0.8660$
$\pi/2$	0	$-\pi/2$	0
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2} \approx 0.8660$	$-\frac{2\pi}{3}$	$-\frac{\sqrt{3}}{2} \approx -0.8660$
$\frac{5\pi}{6}$	$-\frac{\sqrt{3}}{2} \approx -0.8660$	$-\frac{5\pi}{6}$	$\frac{\sqrt{3}}{2} \approx 0.8660$

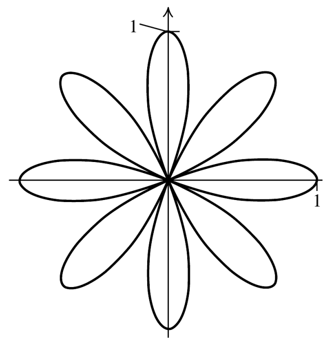
Rose curve



70.

θ	$\cos 4\theta$	θ	$\cos 4\theta$
0	1	π	1
$\frac{\pi}{6}$	$-\frac{1}{2}$	$-\frac{\pi}{6}$	$-\frac{1}{2}$
$\pi/3$	$-\frac{1}{2}$	$-\pi/3$	$-\frac{1}{2}$
$\pi/2$	1	$-\pi/2$	1
$\frac{2\pi}{3}$	$-\frac{1}{2}$	$-\frac{2\pi}{3}$	$-\frac{1}{2}$
$\frac{5\pi}{6}$	$-\frac{1}{2}$	$-\frac{5\pi}{6}$	$-\frac{1}{2}$

Rose curve



71.

θ	$1-2\sin\theta$	θ	$1-2\sin\theta$
0	1	π	1
$\frac{\pi}{6}$	0	$-\frac{\pi}{6}$	2
$\pi/3$	$1-\sqrt{3}$	$-\pi/3$	$1+\sqrt{3}$
$\pi/2$	–1	$-\pi/2$	3
$\frac{2\pi}{3}$	$1-\sqrt{3}$	$-\frac{2\pi}{3}$	$1+\sqrt{3}$
$\frac{5\pi}{6}$	0	$-\frac{5\pi}{6}$	2

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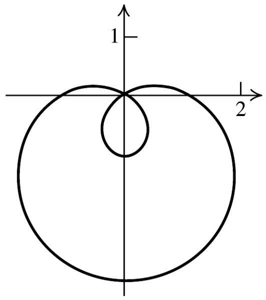
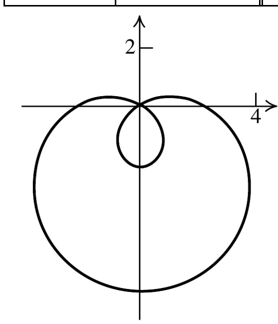


Limaçon

72.

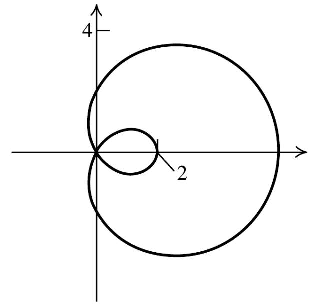
θ	$2-4\sin\theta$	θ	$2-4\sin\theta$
0	2	π	2
$\frac{\pi}{6}$	0	$-\frac{\pi}{6}$	4
$\pi/3$	$2-2\sqrt{3}$	$-\pi/3$	$2+2\sqrt{3}$
$\pi/2$	–2	$-\pi/2$	6
$\frac{2\pi}{3}$	$2-2\sqrt{3}$	$-\frac{2\pi}{3}$	$2+2\sqrt{3}$
$\frac{5\pi}{6}$	0	$-\frac{5\pi}{6}$	4



Limaçon

73.

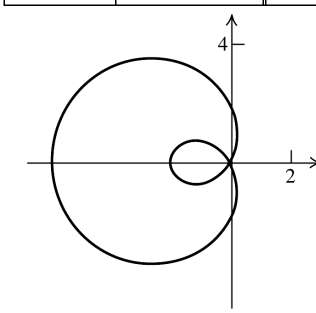
θ	$2+4\cos\theta$	θ	$2+4\cos\theta$
0	6	π	–2
$\frac{\pi}{6}$	$2+2\sqrt{3}$	$-\frac{\pi}{6}$	$2+2\sqrt{3}$
$\pi/3$	4	$-\pi/3$	4
$\pi/2$	2	$-\pi/2$	2
$\frac{2\pi}{3}$	0	$-\frac{2\pi}{3}$	0
$\frac{5\pi}{6}$	$2-2\sqrt{3}$	$-\frac{5\pi}{6}$	$2-2\sqrt{3}$



Limaçon

74.

θ	$2-4\cos\theta$	θ	$2-4\cos\theta$
0	–2	π	6
$\frac{\pi}{6}$	$2-2\sqrt{3}$	$-\frac{\pi}{6}$	$2-2\sqrt{3}$
$\pi/3$	0	$-\pi/3$	0
$\pi/2$	2	$-\pi/2$	2
$\frac{2\pi}{3}$	4	$-\frac{2\pi}{3}$	4
$\frac{5\pi}{6}$	$2+2\sqrt{3}$	$-\frac{5\pi}{6}$	$2+2\sqrt{3}$



Limaçon

75.

θ	$3-2\sin\theta$	θ	$3-2\sin\theta$
0	3	π	3
$\frac{\pi}{6}$	2	$-\frac{\pi}{6}$	4
$\pi/3$	$3-\sqrt{3}$	$-\pi/3$	$3+\sqrt{3}$
$\pi/2$	1	$-\pi/2$	5
$\frac{2\pi}{3}$	$3-\sqrt{3}$	$-\frac{2\pi}{3}$	$3+\sqrt{3}$
$\frac{5\pi}{6}$	2	$-\frac{5\pi}{6}$	4

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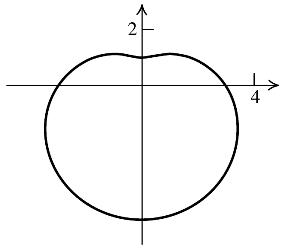
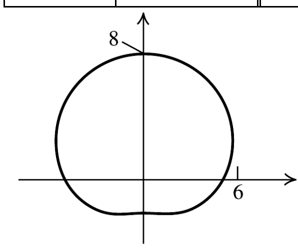


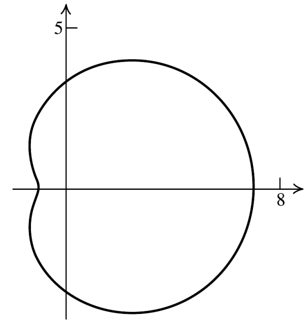
76.

θ	$5+3\sin\theta$	θ	$5+3\sin\theta$
0	5	π	5
$\frac{\pi}{6}$	$\frac{13}{2}$	$-\frac{\pi}{6}$	$\frac{7}{2}$
$\pi/3$	$5+\frac{3\sqrt{3}}{2}$	$-\pi/3$	$5-\frac{3\sqrt{3}}{2}$
$\pi/2$	8	$-\pi/2$	2
$\frac{2\pi}{3}$	$5+\frac{3\sqrt{3}}{2}$	$-\frac{2\pi}{3}$	$5-\frac{3\sqrt{3}}{2}$
$\frac{5\pi}{6}$	$\frac{13}{2}$	$-\frac{5\pi}{6}$	$\frac{7}{2}$



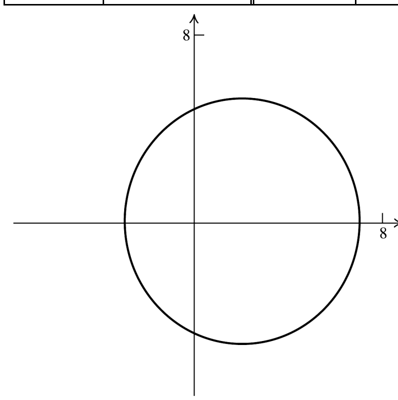
77.

θ	$4+3\cos\theta$	θ	$4+3\cos\theta$
0	7	π	1
$\frac{\pi}{6}$	$4+\frac{3\sqrt{3}}{2}$	$-\frac{\pi}{6}$	$4+\frac{3\sqrt{3}}{2}$
$\pi/3$	$\frac{11}{2}$	$-\pi/3$	$\frac{11}{2}$
$\pi/2$	4	$-\pi/2$	4
$\frac{2\pi}{3}$	$\frac{5}{2}$	$-\frac{2\pi}{3}$	$\frac{5}{2}$
$\frac{5\pi}{6}$	$4-\frac{3\sqrt{3}}{2}$	$-\frac{5\pi}{6}$	$4-\frac{3\sqrt{3}}{2}$



78.

θ	$5+2\cos\theta$	θ	$5+2\cos\theta$
0	7	π	3
$\frac{\pi}{6}$	$5+\sqrt{3}$	$-\frac{\pi}{6}$	$5+\sqrt{3}$
$\pi/3$	6	$-\pi/3$	6
$\pi/2$	5	$-\pi/2$	5
$\frac{2\pi}{3}$	4	$-\frac{2\pi}{3}$	4
$\frac{5\pi}{6}$	$5-\sqrt{3}$	$-\frac{5\pi}{6}$	$5-\sqrt{3}$



7.6 Applying the Concepts

79. $\alpha = 45°, \beta = 30° \Rightarrow$

$\overrightarrow{OE} = \langle 14\cos 45°, 14\sin 45° \rangle = \langle 7\sqrt{2}, 7\sqrt{2} \rangle$

$\overrightarrow{EH} = \langle 8\cos 30°, 8\sin 30° \rangle = \langle 4\sqrt{3}, 4 \rangle$

$\overrightarrow{OH} = \overrightarrow{OE} + \overrightarrow{EH} = \langle 7\sqrt{2}+4\sqrt{3}, 7\sqrt{2}+4 \rangle$

$r = \sqrt{\left(7\sqrt{2}+4\sqrt{3}\right)^2 + \left(7\sqrt{2}+4\right)^2} \approx 21.8$ in.

$\theta = \tan^{-1}\left(\frac{7\sqrt{2}+4}{7\sqrt{2}+4\sqrt{3}}\right) \approx 39.6°$

The hand is at $(21.8, 39.6°)$ relative to the shoulder.

80. $\alpha = -30°, \beta = 60° \Rightarrow \overrightarrow{OE} = \langle 14\cos(-30°), 14\sin(-30°)\rangle = \left\langle 7\sqrt{3}, -7\right\rangle$

$\overrightarrow{EH} = \langle 8\cos 60°, 8\sin 60°\rangle = \left\langle 4, 4\sqrt{3}\right\rangle$

$\overrightarrow{OH} = \overrightarrow{OE} + \overrightarrow{EH} = \left\langle 7\sqrt{3}+4, -7+4\sqrt{3}\right\rangle$

$r = \sqrt{\left(7\sqrt{3}+4\right)^2 + \left(-7+4\sqrt{3}\right)^2} \approx 16.1$ in.

$\theta = \tan^{-1}\left(\dfrac{-7+4\sqrt{3}}{7\sqrt{3}+4}\right) \approx -0.3°$

The hand is at $(16.1, -0.3°)$ relative to the shoulder.

81. $\alpha = -70°, \beta = 0° \Rightarrow \overrightarrow{OE} = \langle 14\cos(-70°), 14\sin(-70°)\rangle$

$\overrightarrow{EH} = \langle 8\cos 0°, 8\sin 0°\rangle = \langle 8, 0\rangle$

$\overrightarrow{OH} = \overrightarrow{OE} + \overrightarrow{EH} = \langle 14\cos(-70°)+8,\ 14\sin(-70°)\rangle$

$r = \sqrt{\left(14\cos(-70°)+8\right)^2 + \left(14\sin(-70)\right)^2} \approx 18.3$ in.

$\theta = \tan^{-1}\left(\dfrac{14\sin(-70)}{14\cos(-70°)+8}\right) \approx -45.8°$

The hand is at $(18.3, -45.8°)$ relative to the shoulder.

82. $\alpha = 47°, \beta = 17° \Rightarrow \overrightarrow{OE} = \langle 14\cos 47°, 14\sin 47°\rangle$

$\overrightarrow{EH} = \langle 8\cos 17°, 8\sin 17°\rangle$

$\overrightarrow{OH} = \overrightarrow{OE} + \overrightarrow{EH} = \langle 14\cos 47° + 8\cos 17°, 14\sin 47° + 8\sin 17°\rangle$

$r = \sqrt{\left(14\cos 47° + 8\cos 17°\right)^2 + \left(14\sin 47° + 8\sin 17°\right)^2} \approx 21.3$ in.

$\theta = \tan^{-1}\left(\dfrac{14\sin 47° + 8\sin 17°}{14\cos 47° + 8\cos 17°}\right) \approx 36.2°$

The hand is at $(21.3, 36.2°)$ relative to the shoulder.

7.6 Beyond the Basics

83. $y = x\tan\left(\sqrt{x^2+y^2}\right) \Rightarrow \dfrac{y}{x} = \tan r \Rightarrow \dfrac{\sin\theta}{\cos\theta} = \tan r \Rightarrow r = \theta$

84. $y = x\tan\left(\ln\sqrt{x^2+y^2}\right) \Rightarrow \dfrac{y}{x} = \tan(\ln r) \Rightarrow \dfrac{\sin\theta}{\cos\theta} = \tan(\ln r) \Rightarrow \ln r = \theta \Rightarrow r = e^{\theta}$

85. $r(1-\sin\theta) = 3 \Rightarrow r - r\sin\theta = 3 \Rightarrow \sqrt{x^2+y^2} - y = 3 \Rightarrow \sqrt{x^2+y^2} = y+3 \Rightarrow$

$x^2+y^2 = (y+3)^2 = y^2+6y+9 \Rightarrow x^2 - 9 = 6y \Rightarrow y = \dfrac{x^2}{6} - \dfrac{3}{2}$

86. $r(1+\cos\theta) = 2 \Rightarrow r + r\cos\theta = 2 \Rightarrow \sqrt{x^2+y^2} + x = 2 \Rightarrow \sqrt{x^2+y^2} = 2 - x \Rightarrow$

$x^2+y^2 = (2-x)^2 = 4 - 4x + x^2 \Rightarrow y^2 = 4 - 4x$

87. $r\left(1+\dfrac{1}{2}\cos\theta\right) = 1 \Rightarrow r + \dfrac{r\cos\theta}{2} = 1 \Rightarrow \sqrt{x^2+y^2} + \dfrac{x}{2} = 1 \Rightarrow \sqrt{x^2+y^2} = 1 - \dfrac{x}{2} \Rightarrow$

$x^2+y^2 = \left(1-\dfrac{x}{2}\right)^2 = 1 - x + \dfrac{x^2}{4} \Rightarrow y^2 = 1 - x - \dfrac{3x^2}{4}$

88. $r\left(1+\frac{3}{4}\sin\theta\right)=3 \Rightarrow r+\frac{3r\sin\theta}{4}=3 \Rightarrow \sqrt{x^2+y^2}+\frac{3y}{4}=3 \Rightarrow \sqrt{x^2+y^2}=3-\frac{3y}{4} \Rightarrow$

$x^2+y^2=\left(3-\frac{3y}{4}\right)^2=9-\frac{9y}{2}+\frac{9y^2}{16} \Rightarrow x^2+\frac{7y^2}{16}+\frac{9y}{2}=9$

Multiply both sides by $\frac{16}{7}$, and then complete the square on *y*.

$x^2+\frac{7y^2}{16}+\frac{9y}{2}=9 \Rightarrow \frac{16x^2}{7}+y^2+\frac{72y}{7}+\frac{1296}{49}=\frac{144}{7}+\frac{1296}{49} \Rightarrow \frac{16x^2}{7}+\left(y+\frac{36}{7}\right)^2=\frac{2304}{49}$

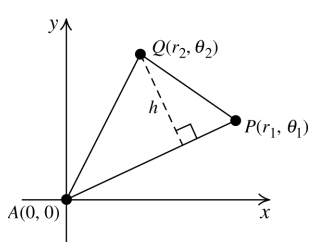
89. $r(1-3\cos\theta)=5 \Rightarrow r-3r\cos\theta=5 \Rightarrow \sqrt{x^2+y^2}-3x=5 \Rightarrow \sqrt{x^2+y^2}=3x+5 \Rightarrow$
$x^2+y^2=(3x+5)^2=9x^2+30x+25 \Rightarrow y^2-8x^2-30x-25=0$

90. $r(1-2\sin\theta)=4 \Rightarrow r-2r\sin\theta=4 \Rightarrow \sqrt{x^2+y^2}-2y=4 \Rightarrow \sqrt{x^2+y^2}=2y+4 \Rightarrow$
$x^2+y^2=(2y+4)^2=4y^2+16y+16 \Rightarrow x^2-3y^2-16y-16=0$

91. $D=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{(r_2\cos\theta_2-r_1\cos\theta_1)^2+(r_2\sin\theta_2-r_1\sin\theta_1)^2}$
$=\sqrt{r_2{}^2\cos^2\theta_2-2r_1r_2\cos\theta_1\cos\theta_2+r_1{}^2\cos^2\theta_1+r_2{}^2\sin^2\theta_2-2r_1r_2\sin\theta_1\sin\theta_2+r_1{}^2\sin^2\theta_1}$
$=\sqrt{r_2{}^2\left(\cos^2\theta_2+\sin^2\theta_2\right)-2r_1r_2\left(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\right)+r_1{}^2\left(\cos^2\theta_1+\sin^2\theta_1\right)}$
$=\sqrt{r_1{}^2+r_2{}^2-2r_1r_2\left(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\right)}$
$=\sqrt{r_1{}^2+r_2{}^2-2r_1r_2\cos\left(\theta_1-\theta_2\right)}$

92. $r=a\sin\theta+b\cos\theta \Rightarrow r=\frac{ay}{r}+\frac{bx}{r} \Rightarrow r^2=ay+bx \Rightarrow x^2+y^2=ay+bx \Rightarrow x^2-bx+y^2-ay=0 \Rightarrow$
$x^2-bx+\frac{b^2}{4}+y^2-ay+\frac{a^2}{4}=\frac{b^2}{4}+\frac{a^2}{4} \Rightarrow \left(x-\frac{b}{2}\right)^2+\left(y-\frac{a}{2}\right)^2=\frac{b^2}{4}+\frac{a^2}{4}.$
This is the equation of a circle with center $\left(\frac{b}{2},\frac{a}{2}\right)$ and radius $\sqrt{\frac{b^2}{4}+\frac{a^2}{4}}$.

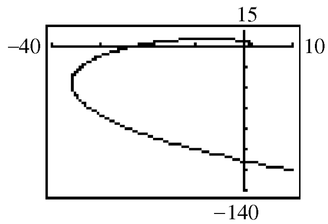
93.



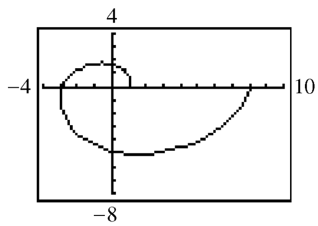
$h=d(A,Q)\sin A, d(A,Q)=r_2, d(A,P)=r_1$, and $m\angle A=\theta_2-\theta_1$. So, $k=\frac{1}{2}r_1r_2\sin\left(\theta_2-\theta_1\right)$.

94. $r=2a\cos\left(\theta-\theta_0\right) \Rightarrow r=2a(\cos\theta\cos\theta_0+\sin\theta\sin\theta_0)=2a\cos\theta_0\cos\theta+2a\sin\theta_0\sin\theta.$
From exercise 74, we know that $r=A\sin\theta+B\cos\theta$ is a circle. $A=2a\cos\theta_0$ and $B=2a\sin\theta_0$, so the equation is a circle with center $\left(a\cos\theta_0, a\sin\theta_0\right)$ or $\left(a,\theta_0\right)$ in polar coordinates.
When $\theta_0=0$, the circle is centered at $(a,0)$. When $\theta_0=0$, the circle is centered at $(0,a)$.

95.



96.



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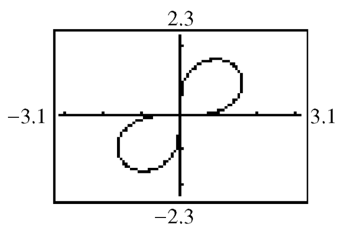

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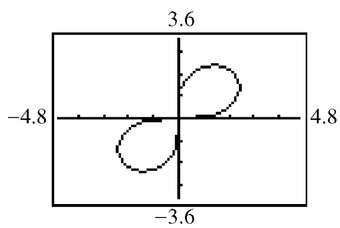

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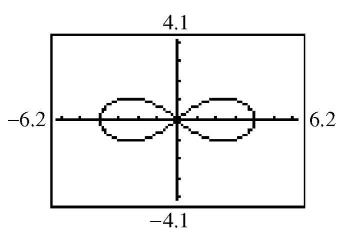
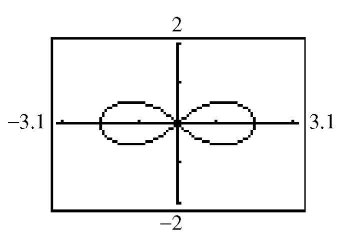


100.



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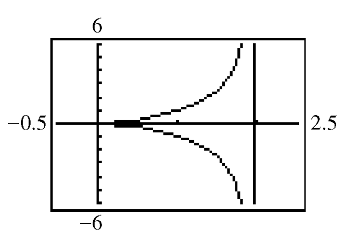

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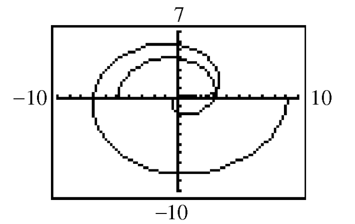

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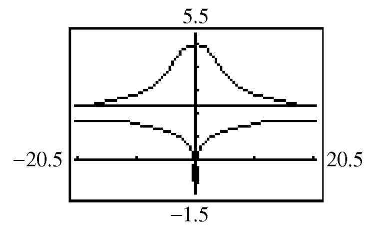
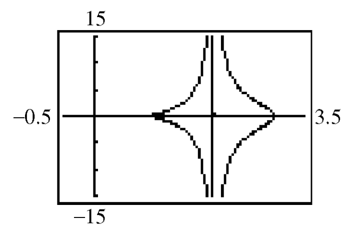
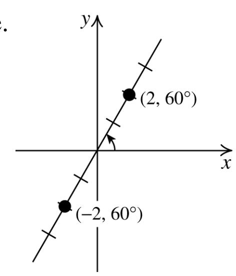


104.



7.6 Critical Thinking/Discussion/Writing

105. False.



106. True **107.** True **108.** True

7.6 Maintaining Skills

109. $(2-3i)+(-5+i)=-3-2i$

110. $(5+3i)-(4-2i)=1+5i$

111. $i(2+5i)=2i+5i^2=2i-5=-5+2i$

112. $-i(5-3i)=-5i+3i^2=-5i-3=-3-5i$

113.
$$\begin{aligned}(2+i)(3-i)&=6+i-i^2=6+i-(-1)\\&=7+i\end{aligned}$$

114.
$$\begin{aligned}(4-3i)(2-5i)&=8-26i+15i^2=8-26i-15\\&=-7-26i\end{aligned}$$

115.
$$\begin{aligned}(2-3i)^2&=4-12i+9i^2=4-12i-9\\&=-5-12i\end{aligned}$$

116. $(1+2i)^3 = 1+6i+12i^2+8i^3$
$= 1+6i-12-8i = -11-2i$

117. $\frac{1}{3-4i} = \frac{1}{3-4i}\cdot\frac{3+4i}{3+4i} = \frac{3+4i}{9-16i^2} = \frac{3+4i}{9+16}$
$= \frac{3+4i}{25} = \frac{3}{25}+\frac{4}{25}i$

118. $\frac{2i}{2+3i} = \frac{2i}{2+3i}\cdot\frac{2-3i}{2-3i} = \frac{4i-6i^2}{4-9i^2} = \frac{4i+6}{4+9}$
$= \frac{6+4i}{13} = \frac{6}{13}+\frac{4}{13}i$

119. $\frac{2-3i}{3+4i} = \frac{2-3i}{3+4i}\cdot\frac{3-4i}{3-4i} = \frac{6-17i+12i^2}{9-16i^2}$
$= \frac{6-17i-12}{9+16} = \frac{-6-17i}{25} = -\frac{6}{25}-\frac{17}{25}i$

120. $\frac{1+i}{2-i} = \frac{1+i}{2-i}\cdot\frac{2+i}{2+i} = \frac{2+3i+i^2}{4-i^2} = \frac{2+3i-1}{4-(-1)}$
$= \frac{1+3i}{5} = \frac{1}{5}+\frac{3}{5}i$

121. $\mathbf{v} = \mathbf{i}+\mathbf{j}$
$\|\mathbf{v}\| = \sqrt{1^2+1^2} = \sqrt{2}$
$\theta' = \left|\tan^{-1}\left(\frac{1}{1}\right)\right| = \left|\frac{\pi}{4}\right| = \frac{\pi}{4}$
Since (1, 1) lies in quadrant I, we have
$\theta = \frac{\pi}{4}$. Thus,
$\mathbf{v} = \mathbf{i}+\mathbf{j} = \sqrt{2}\left(\cos\frac{\pi}{4}\mathbf{i}+\sin\frac{\pi}{4}\mathbf{j}\right)$

122. $\mathbf{v} = 2\mathbf{i}-2\mathbf{j}$
$\|\mathbf{v}\| = \sqrt{2^2+(-2)^2} = \sqrt{8} = 2\sqrt{2}$
$\theta' = \left|\tan^{-1}\left(\frac{-2}{2}\right)\right| = \left|\tan^{-1}(-1)\right| = \left|-\frac{\pi}{4}\right| = \frac{\pi}{4}$
Since (2, −2) lies in quadrant IV, we have
$\theta = 2\pi-\frac{\pi}{4} = \frac{7\pi}{4}$. Thus,
$\mathbf{v} = 2\mathbf{i}-2\mathbf{j} = 2\sqrt{2}\left(\cos\frac{7\pi}{4}\mathbf{i}-\sin\frac{7\pi}{4}\mathbf{j}\right)$

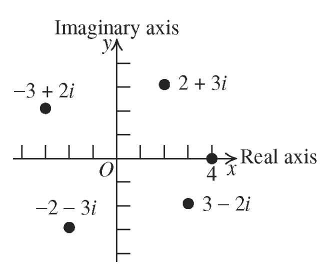
123. $\mathbf{v} = -\sqrt{3}\mathbf{i}+\mathbf{j}$
$\|\mathbf{v}\| = \sqrt{\left(-\sqrt{3}\right)^2+1^2} = 2$
$\theta' = \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) = \left|\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)\right|$
$= \left|-\frac{\pi}{6}\right| = \frac{\pi}{6}$
Since $\left(-\sqrt{3}, 1\right)$ lies in quadrant II, we have
$\theta = \pi-\frac{\pi}{6} = \frac{5\pi}{6}$. Thus,
$\mathbf{v} = -\sqrt{3}\mathbf{i}+\mathbf{j} = 2\left(\cos\frac{5\pi}{6}\mathbf{i}+\sin\frac{5\pi}{6}\mathbf{j}\right).$

124. $\mathbf{v} = -\mathbf{i}-\sqrt{3}\mathbf{j}$
$\|\mathbf{v}\| = \sqrt{(-1)^2+\left(-\sqrt{3}\right)^2} = 2$
$\theta' = \tan^{-1}\left(\frac{-\sqrt{3}}{-1}\right) = \left|\tan^{-1}\left(\sqrt{3}\right)\right| = \left|\frac{\pi}{3}\right| = \frac{\pi}{3}$
Since $\left(-1, -\sqrt{3}\right)$ lies in quadrant III, we have
$\theta = \pi+\frac{\pi}{3} = \frac{4\pi}{3}$. Thus,
$\mathbf{v} = -\mathbf{i}-\sqrt{3}\mathbf{j} = 2\left(\cos\frac{4\pi}{3}\mathbf{i}+\sin\frac{4\pi}{3}\mathbf{j}\right).$

7.7 Polar Form of Complex Numbers; DeMoivre's Theorem

7.7 Practice Problems

1.



2. a. $|-5+12i| = \sqrt{(-5)^2+12^2} = 13$

b. $|-7| = 7$

c. $|i| = 1$

d. $|a-bi| = \sqrt{a^2+(-b)^2} = \sqrt{a^2+b^2}$

3. $z=-1-i \Rightarrow a=-1,\ b=-1$

$r=\sqrt{(-1)^2+(-1)^2}=\sqrt{2}$

$\theta=\tan^{-1}\left(\frac{-1}{-1}\right)=\tan^{-1}1 \Rightarrow \theta=\frac{\pi}{4}$ or $\theta=\frac{5\pi}{4}$

Since (−1, −1) lies in quadrant III, $\theta=\frac{5\pi}{4}$.

Thus, $z=-1-i=\sqrt{2}\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)$.

4. $z=4\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)=4\cos\frac{5\pi}{3}+4i\sin\frac{5\pi}{3}$

$=4\left(\frac{1}{2}\right)+4i\left(-\frac{\sqrt{3}}{2}\right)=2-2i\sqrt{3}$

5. $z_1=5(\cos 75°+i\sin 75°)$

$z_2=2(\cos 60°+i\sin 60°)$

$z_1z_2=5\cdot 2\left(\cos(75°+60°)+i\sin(75°+60°)\right)$

$=10(\cos 135°+i\sin 135°)$

$\frac{z_1}{z_2}=\frac{5}{2}\left(\cos(75°-60°)+i\sin(75°-60°)\right)$

$=\frac{5}{2}(\cos 15°+i\sin 15°)$

6. $Z_1=4\left(\cos 45°+i\sin 45°\right)$

$Z_2=6\left(\cos 0°+i\sin 0°\right)$

To find Z_1+Z_2, first convert each to rectangular form. Then convert the sum back to polar form:

$Z_1=4\left(\cos 45°+i\sin 45°\right)=4\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)$

$=2\sqrt{2}+2\sqrt{2}i$

$Z_2=6\left(\cos 0°+i\sin 0°\right)=6$

$Z_1+Z_2=6+2\sqrt{2}+2\sqrt{2}i \Rightarrow$

$a=6+2\sqrt{2}, b=2\sqrt{2}$

$r=\sqrt{\left(6+2\sqrt{2}\right)^2+\left(2\sqrt{2}\right)^2}\approx 9.27$

$\tan\theta=\frac{2\sqrt{2}}{6+2\sqrt{2}}$

$\theta=\tan^{-1}\left(\frac{2\sqrt{2}}{6+2\sqrt{2}}\right)\approx 17.8°$

The polar form of Z_1+Z_2 is approximately $9.27\left(\cos 17.8°+i\sin 17.8°\right)$.

Now find Z_1Z_2.

$Z_1Z_2=4\cdot 6\left(\cos\left(45°+0°\right)+i\sin\left(45°+0°\right)\right)$

$=24\left(\cos 45°+i\sin 45°\right)$

$\frac{Z_1Z_2}{Z_1+Z_2}\approx\frac{24\left(\cos 45°+i\sin 45°\right)}{9.27\left(\cos 17.8°+i\sin 17.8°\right)}$

$\approx 2.6\left(\cos\left(45°-17.8°\right)+i\sin\left(45°-17.8°\right)\right)$

$\approx 2.6\left(\cos 27.2°+i\sin 27.2°\right)$

7. First convert $-1+i$ to polar form:

$r=\sqrt{(-1)^2+1^2}=\sqrt{2};\theta=\tan^{-1}(-1)=\frac{3\pi}{4}$

(Note that $-1+i$ lies in Quadrant II.)

a. $(-1+i)^8=\left[\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)\right]^8$

$=\left(\sqrt{2}\right)^8\left(\cos\left(8\cdot\frac{3\pi}{4}\right)+i\sin\left(8\cdot\frac{3\pi}{4}\right)\right)$

$=2^4\left(\cos 6\pi+i\sin 6\pi\right)$

$=16\cdot 1+16\cdot 0i=16$

b. $(-1+i)^{-12}$

$=\left[\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)\right]^{-12}$

$=\left(\sqrt{2}\right)^{-12}\left(\cos\left(-12\cdot\frac{3\pi}{4}\right)+i\sin\left(-12\cdot\frac{3\pi}{4}\right)\right)$

$=\frac{1}{2^6}\left(\cos\left(-9\pi\right)+i\sin\left(-9\pi\right)\right)$

$=\frac{1}{64}(-1)+\frac{1}{64}i\cdot 0=-\frac{1}{64}$

8. First convert $-1+i$ to polar form:

$r=\sqrt{(-1)^2+1^2}=\sqrt{2};\theta=\tan^{-1}(-1)=135°$

(Note that $-1+i$ lies in Quadrant II.)

$z_k=\left(\sqrt{2}\right)^{1/3}\left(\cos\frac{135°+360°\cdot k}{3}+i\sin\frac{135°+360°\cdot k}{3}\right)$

for k = 0, 1, 2.

$z_0=\left(\sqrt{2}\right)^{1/3}\left(\cos\frac{135°}{3}+i\sin\frac{135°}{3}\right)$

$=2^{1/6}\left(\cos 45°+i\sin 45°\right)$

$z_1=\left(\sqrt{2}\right)^{1/3}\left(\cos\frac{495°}{3}+i\sin\frac{495°}{3}\right)$

$=2^{1/6}\left(\cos 165°+i\sin 165°\right)$

$z_2=\left(\sqrt{2}\right)^{1/3}\left(\cos\frac{855°}{3}+i\sin\frac{855°}{3}\right)$

$=2^{1/6}\left(\cos 285°+i\sin 285°\right)$

9. The polar form for 1 is
$1+0i=\cos 0°+i\sin 0°$
$z_k=\cos\frac{0°+360°\cdot k}{4}+i\sin\frac{0°+360°\cdot k}{4}$ for $k=0, 1, 2, 3.$
$z_0=\cos\frac{0°}{4}+i\sin\frac{0°}{4}=\cos 0°+i\sin 0°$
$z_1=\cos\frac{360°}{4}+i\sin\frac{360°}{4}=\cos 90°+i\sin 90°$
$z_2=\cos\frac{720°}{4}+i\sin\frac{720°}{4}$
$=\cos 180°+i\sin 180°$
$z_3=\cos\frac{1080°}{4}+i\sin\frac{1080°}{4}$
$=\cos 270°+i\sin 270°$

7.7 Basic Concepts and Skills

1. If $z=a+bi$, then $|z|=\underline{\sqrt{a^2+b^2}}$ and $\arg z=\theta=\underline{\tan^{-1}\left(\frac{b}{a}\right)}$.

2. If $|z|=r$ and $\arg z=\theta$, then the polar form of z is $z=\underline{r(\cos\theta+i\sin\theta)}$.

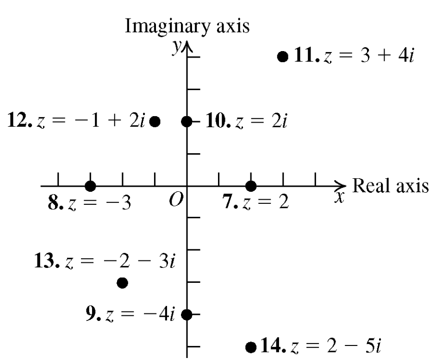
3. To multiply two complex numbers in polar form, multiply their <u>moduli</u> and <u>add</u> their arguments.

4. DeMoivre's Theorem states that $\left[r(\cos\theta+i\sin\theta)\right]^n=\underline{r^n(\cos n\theta+i\sin n\theta)}$.

5. True.
$z^{-1}=r^{-1}(\cos(-\theta)+i\sin(-\theta))$
$=\frac{1}{r}(\cos\theta-i\sin\theta)$

6. True

7–14.



7. $|2|=2$

8. $|-3|=3$

9. $|-4i|=\sqrt{(-4)^2}=4$

10. $|2i|=\sqrt{2^2}=2$

11. $|3+4i|=\sqrt{3^2+4^2}=5$

12. $|-1+2i|=\sqrt{(-1)^2+2^2}=\sqrt{5}$

13. $|-2-3i|=\sqrt{(-2)^2+(-3)^2}=\sqrt{13}$

14. $|2-5i|=\sqrt{2^2+(-5)^2}=\sqrt{29}$

15. $r=\sqrt{1^2+\left(\sqrt{3}\right)^2}=2;\theta=\tan^{-1}\left(\frac{\sqrt{3}}{1}\right)=60°$ or $\theta=240°$. $1+\sqrt{3}i$ is in Quadrant I, so $\theta=60°$
$1+\sqrt{3}i=2(\cos 60°+i\sin 60°)$

16. $r=\sqrt{(-1)^2+\left(\sqrt{2}\right)^2}=\sqrt{3}$
$\theta=\tan^{-1}\left(-\frac{\sqrt{2}}{1}\right)\approx 125.3°$ or $\theta\approx 305.3°$
$-1+\sqrt{2}i$ is in Quadrant II, so $\theta\approx 125.3°$
$-1+\sqrt{2}i=\sqrt{3}(\cos 125.3°+i\sin 125.3°)$

17. $r=\sqrt{(-1)^2+1^2}=\sqrt{2};\theta=\tan^{-1}(-1)=135°$ or $\theta=315°$. Note that $-1+i$ lies in Quadrant II, so $\theta=135°$
$-1+i=\sqrt{2}(\cos 135°+i\sin 135°)$

18. $r=\sqrt{1^2+(-1)^2}=\sqrt{2};\theta=\tan^{-1}(-1)=135°$ or $\theta=315°$. Note that $1-i$ lies in Quadrant III, so $\theta=315°$.
$1-i=\sqrt{2}(\cos 315°+i\sin 315°)$

19. $r=\sqrt{0^2+1^2}=1;\tan\theta=\frac{1}{0}\Rightarrow\tan\theta$ is undefined $\Rightarrow\theta=90°$ or $\theta=270°$
$i=\cos 90°+i\sin 90°$

20. $r=\sqrt{0^2+(-1)^2}=1;\tan\theta=-\frac{1}{0}\Rightarrow\tan\theta$ is undefined $\Rightarrow\theta=90°$ or $\theta=270°$
$-i=\cos 270°+i\sin 270°$

21. $r=\sqrt{1^2+0^2}=1;\theta=\tan^{-1}0=0°$ or $\theta=180°$
$i=\cos 0°+i\sin 0°$

22. $r=\sqrt{(-1)^2+0^2}=1;\theta=\tan^{-1}0=0°$ or $\theta=180°$
$i=\cos 180°+i\sin 180°$

23. $r = \sqrt{3^2 + (-3)^2} = 3\sqrt{2}$
$\theta = \tan^{-1}\left(-\frac{3}{3}\right) = \tan^{-1}(-1) = 135°$ or $\theta = 315°$
$3 - 3i$ is in Quadrant IV, so $\theta = 315°$
$3 - 3i = 3\sqrt{2}(\cos 315° + i \sin 315°)$

24. $r = \sqrt{\left(4\sqrt{3}\right)^2 + 4^2} = 8$
$\theta = \tan^{-1} \frac{4}{4\sqrt{3}} = \tan^{-1} \frac{\sqrt{3}}{3} = 30°$ or$\theta = 210°$
$4\sqrt{3} + 4i$ is in Quadrant I, so $\theta = 30°$
$4\sqrt{3} + 4i = 8(\cos 30° + i \sin 30°)$

25. $r = \sqrt{2^2 + \left(-2\sqrt{3}\right)^2} = 4$
$\theta = \tan^{-1}\left(-\frac{2\sqrt{3}}{2}\right) = 120°$ or $\theta = 300°$.
$2 - 2\sqrt{3}i$ is in Quadrant IV, so $\theta = 300°$.
$2 - 2\sqrt{3}i = 4(\cos 300° + i \sin 300°)$

26. $r = \sqrt{2^2 + 3^2} = \sqrt{13}; \theta = \tan^{-1}\left(\frac{3}{2}\right) \approx 56.3°$ or
$\theta = 236.3°$.
$2 + 3i$ is in Quadrant I, so $\theta \approx 56.3°$
$2 + 3i = \sqrt{13}(\cos 56.3° + i \sin 56.3°)$

27. $2(\cos 60° + i \sin 60°) = 2\cos 60° + 2i \sin 60°$
$= 1 + \sqrt{3}i$

28. $4(\cos 120° + i \sin 120°) = 4\cos 120° + 4i \sin 120°$
$= -2 + 2\sqrt{3}i$

29. $3(\cos \pi + i \sin \pi) = 3\cos \pi + 3i \sin \pi = -3$

30. $5\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right) = 5\cos \frac{\pi}{2} + 5i \sin \frac{\pi}{2} = 5i$

31. $5(\cos 240° + i \sin 240°) = 5\cos 240° + 5i \sin 240°$
$= -\frac{5}{2} - \frac{5\sqrt{3}}{2}i$

32. $2(\cos 300° + i \sin 300°) = 2\cos 300° + 2i \sin 300°$
$= 1 - \sqrt{3}i$

33. $8(\cos 0° + i \sin 0°) = 8\cos 0° + 8i \sin 0° = 8$

34. $2(\cos(-90°) + i \sin(-90°))$
$= 2\cos(-90°) + 2i \sin(-90°) = -2i$

35. $6\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right) = 6\cos \frac{5\pi}{6} + 6i \sin \frac{5\pi}{6}$
$= -3\sqrt{3} + 3i$

36. $4\left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right) = 4\cos \frac{3\pi}{4} + 4i \sin \frac{3\pi}{4}$
$= -2\sqrt{2} + 2\sqrt{2}i$

37. $3\left(\cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right)\right)$
$= 3\cos\left(-\frac{\pi}{3}\right) + 3i \sin\left(-\frac{\pi}{3}\right) = \frac{3}{2} - \frac{3\sqrt{3}}{2}i$

38. $5\left(\cos\left(-\frac{7\pi}{6}\right) + i \sin\left(-\frac{7\pi}{6}\right)\right)$
$= 5\cos\left(-\frac{7\pi}{6}\right) + 5i \sin\left(-\frac{7\pi}{6}\right) = -\frac{5\sqrt{3}}{2} + \frac{5}{2}i$

39. $z_1 = 4(\cos 75° + i \sin 75°)$
$z_2 = 2(\cos 15° + i \sin 15°)$
$z_1 z_2 = 4 \cdot 2(\cos(75° + 15°) + i \sin(75° + 15°))$
$= 8(\cos 90° + i \sin 90°)$
$\frac{z_1}{z_2} = \frac{4}{2}(\cos(75° - 15°) + i \sin(75° - 15°))$
$= 2(\cos 60° + i \sin 60°)$

40. $z_1 = 6(\cos 90° + i \sin 90°)$
$z_2 = 2(\cos 45° + i \sin 45°)$
$z_1 z_2 = 6 \cdot 2(\cos(90° + 45°) + i \sin(90° + 45°))$
$= 12(\cos 135° + i \sin 135°)$
$\frac{z_1}{z_2} = \frac{6}{2}(\cos(90° - 45°) + i \sin(90° - 45°))$
$= 3(\cos 45° + i \sin 45°)$

41. $z_1 = 5(\cos 240° + i \sin 240°)$
$z_2 = 2(\cos 60° + i \sin 60°)$
$z_1 z_2 = 5 \cdot 2(\cos(240° + 60°) + i \sin(240° + 60°))$
$= 10(\cos 300° + i \sin 300°)$
$\frac{z_1}{z_2} = \frac{5}{2}(\cos(240° - 60°) + i \sin(240° - 60°))$
$= \frac{5}{2}(\cos 180° + i \sin 180°)$

42. $z_1 = 10(\cos 135° + i \sin 135°)$
$z_2 = 4(\cos 225° + i \sin 225°)$
$z_1 z_2 = 10 \cdot 4(\cos(135° + 225°) + i \sin(240° + 60°))$
$= 40(\cos 360° + i \sin 360°)$
$= 40(\cos 0° + i \sin 0°)$
$\frac{z_1}{z_2} = \frac{5}{2}(\cos(135° - 225°) + i \sin(135° - 225°))$
$= \frac{5}{2}(\cos(-90°) + i \sin(-90°))$
$= \frac{5}{2}(\cos 270° + i \sin 270°)$

43. $z_1 = 3(\cos 40° + i\sin 40°)$
$z_2 = 5(\cos 20° + i\sin 20°)$
$z_1 z_2 = 3\cdot 5(\cos(40°+20°) + i\sin(40°+20°))$
$= 15(\cos 60° + i\sin 60°)$
$\frac{z_1}{z_2} = \frac{3}{5}(\cos(40°-20°) + i\sin(40°-20°))$
$= \frac{3}{5}(\cos 20° + i\sin 20°)$

44. $z_1 = 5(\cos 65° + i\sin 65°)$
$z_2 = 2(\cos 25° + i\sin 25°)$
$z_1 z_2 = 5\cdot 2(\cos(65°+25°) + i\sin(65°+25°))$
$= 10(\cos 90° + i\sin 90°)$
$\frac{z_1}{z_2} = \frac{5}{2}(\cos(65°-25°) + i\sin(65°-25°))$
$= \frac{5}{2}(\cos 40° + i\sin 40°)$

45. $z_1 = 1 + i \Rightarrow r = \sqrt{2}; \theta = \tan^{-1} 1 = 45°$
$z_1 = \sqrt{2}(\cos 45° + i\sin 45°)$
$z_2 = 1 - i \Rightarrow r = \sqrt{2}; \theta = \tan^{-1}(-1) = 315°$
$z_2 = \sqrt{2}(\cos 315° + i\sin 315°)$
$z_1 z_2 = \sqrt{2}\cdot\sqrt{2}(\cos(45°+315°) + i\sin(45°+315°))$
$= 2(\cos 360° + i\sin 360°)$
$= 2(\cos 0° + i\sin 0°)$
$\frac{z_1}{z_2} = \frac{\sqrt{2}}{\sqrt{2}}(\cos(45°-315°) + i\sin(45°-315°))$
$= \cos(-270°) + i\sin(-270°)$
$= \cos 90° + i\sin 90°$

46. $z_1 = 1 + \sqrt{3}i \Rightarrow r = 2; \theta = \tan^{-1}\sqrt{3} = 60°$
$z_1 = 2(\cos 60° + i\sin 60°)$
$z_2 = 1 + i \Rightarrow r = \sqrt{2}; \theta = \tan^{-1} 1 = 45°$
$z_2 = \sqrt{2}(\cos 45° + i\sin 45°)$
$z_1 z_2 = 2\sqrt{2}(\cos(60°+45°) + i\sin(60°+45°))$
$= 2\sqrt{2}(\cos 105° + i\sin 105°)$
$\frac{z_1}{z_2} = \frac{2}{\sqrt{2}}(\cos(60°-45°) + i\sin(60°-45°))$
$= \sqrt{2}(\cos 15° + i\sin 15°)$

47. $z_1 = -\sqrt{3} + i \Rightarrow r = 2; \theta = \tan^{-1} - \frac{\sqrt{3}}{3} = 150°$
$z_1 = 2(\cos 150° + i\sin 150°)$
$z_2 = 2 + 2i \Rightarrow r = 2\sqrt{2}; \theta = \tan^{-1} 1 = 45°$
$z_2 = 2\sqrt{2}(\cos 45° + i\sin 45°)$
$z_1 z_2 = 4\sqrt{2}(\cos(150°+45°) + i\sin(150°+45°))$
$= 4\sqrt{2}(\cos 195° + i\sin 195°)$
$\frac{z_1}{z_2} = \frac{2}{2\sqrt{2}}(\cos(150°-45°) + i\sin(150°-45°))$
$= \frac{\sqrt{2}}{2}(\cos 105° + i\sin 105°)$

48. $z_1 = 3 + 3i \Rightarrow r = 3\sqrt{2}; \theta = \tan^{-1} 1 = 45°$
$z_1 = 3\sqrt{2}(\cos 45° + i\sin 45°)$
$z_2 = 2 - 2\sqrt{3}i \Rightarrow r = 4; \theta = \tan^{-1}\left(-\sqrt{3}\right) = 300°$
$z_2 = 4(\cos 300° + i\sin 300°)$
$z_1 z_2 = 12\sqrt{2}(\cos(45°+300°) + i\sin(45°+300°))$
$= 12\sqrt{2}(\cos 345° + i\sin 345°)$
$\frac{z_1}{z_2} = \frac{3\sqrt{2}}{4}(\cos(45°-300°) + i\sin(45°-300°))$
$= \frac{3\sqrt{2}}{4}(\cos(-255°) + i\sin(-255°))$
$= \frac{3\sqrt{2}}{4}(\cos 105° + i\sin 105°)$

49. $z_1 = 2\sqrt{3} + 2i \Rightarrow r = 4; \theta = \tan^{-1}\frac{\sqrt{3}}{3} = 30°$
$z_1 = 4(\cos 30° + i\sin 30°)$
$z_2 = \sqrt{3} - i \Rightarrow r = 2; \theta = \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right) = 330°$
$z_2 = 2(\cos 330° + i\sin 330°)$
$z_1 z_2 = 4\cdot 2\left(\cos(30°+330°) + i\sin(30°+330°)\right)$
$= 8(\cos 360° + i\sin 360°) = 8$
$\frac{z_1}{z_2} = \frac{4(\cos 30° + i\sin 30°)}{2(\cos 330° + i\sin 330°)}$
$= 2\left(\cos\left(30°-330°\right) + i\sin\left(30°-330°\right)\right)$
$= 2\left(\cos\left(-300°\right) + i\sin\left(-300°\right)\right)$
$= 2\left(\cos 60° + i\sin 60°\right)$

50. $z_1 = 4\sqrt{3} + 4i \Rightarrow r = 8; \theta = \tan^{-1}\frac{\sqrt{3}}{3} = 30°$
$z_1 = 8(\cos 30° + i\sin 30°)$
$z_2 = 3 - 3i \Rightarrow r = 3\sqrt{2}; \theta = \tan^{-1}\left(-1\right) = 315°$
$z_2 = 3\sqrt{2}(\cos 315° + i\sin 315°)$
$z_1 z_2 = 24\sqrt{2}(\cos(30°+315°) + i\sin(30°+315°))$
$= 24\sqrt{2}(\cos 345° + i\sin 345°)$
$\frac{z_1}{z_2} = \frac{4\sqrt{2}}{3}(\cos(30°-315°) + i\sin(30°-315°))$
$= \frac{4\sqrt{2}}{3}(\cos(-285°) + i\sin(-285°))$
$= \frac{4\sqrt{2}}{3}(\cos 75° + i\sin 75°)$

51. $\left[2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\right]^{12}$
$=2^{12}\left(\cos 12\left(\frac{\pi}{3}\right)+i\sin 12\left(\frac{\pi}{3}\right)\right)$
$=2^{12}(\cos 4\pi+i\sin 4\pi)=2^{12}$

52. $\left[\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]^{20}$
$=\sqrt{2}^{20}\left(\cos 20\left(\frac{\pi}{4}\right)+i\sin 20\left(\frac{\pi}{4}\right)\right)$
$=2^{10}(\cos 5\pi+i\sin 5\pi)=-2^{10}$

53. $\left[2\left(\cos\left(-\frac{3\pi}{4}\right)+i\sin\left(-\frac{3\pi}{4}\right)\right)\right]^{6}$
$=2^{6}\left(\cos 6\left(-\frac{3\pi}{4}\right)+i\sin 6\left(-\frac{3\pi}{4}\right)\right)$
$=2^{6}\left(\cos\left(-\frac{9\pi}{2}\right)+i\sin\left(-\frac{9\pi}{2}\right)\right)$
$=-2^{6}i=-64i$

54. $\left(\cos\frac{\pi}{27}+i\sin\frac{\pi}{27}\right)^{-9}$
$=\cos\left(-9\left(\frac{\pi}{27}\right)\right)+i\sin\left(-9\left(\frac{\pi}{27}\right)\right)$
$=\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)=\frac{1}{2}-\frac{\sqrt{3}}{2}i$

55. $\left[2\left(\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)\right)\right]^{-6}$
$=2^{-6}\left(\cos\left((-6)\left(\frac{3\pi}{4}\right)\right)+i\sin\left((-6)\left(\frac{3\pi}{4}\right)\right)\right)$
$=2^{-6}\left(\cos\left(-\frac{9\pi}{2}\right)+i\sin\left(-\frac{9\pi}{2}\right)\right)$
$=-2^{-6}i=-\frac{1}{64}i$

56. $\left[3\left(\cos\left(-\frac{5\pi}{6}\right)+i\sin\left(-\frac{5\pi}{6}\right)\right)\right]^{16}$
$=3^{16}\left(\cos 16\left(-\frac{5\pi}{6}\right)+i\sin 16\left(-\frac{5\pi}{6}\right)\right)$
$=3^{16}\left(\cos\left(-\frac{40}{3}\pi\right)+i\sin\left(-\frac{40}{3}\pi\right)\right)$
$=3^{16}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$

57. $\left[2\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right)\right]^{-10}$
$=2^{-10}\left(\cos\left(-10\left(-\frac{\pi}{4}\right)\right)+i\sin\left(-10\left(-\frac{\pi}{4}\right)\right)\right)$
$=2^{-10}\left(\cos\frac{5\pi}{2}+i\sin\frac{5\pi}{2}\right)=\frac{1}{2^{10}}i$

58. $\left[\frac{1}{2}\left(\cos\left(-\frac{3\pi}{4}\right)+i\sin\left(-\frac{3\pi}{4}\right)\right)\right]^{-6}$
$=\left(\frac{1}{2}\right)^{-6}\left(\cos\left((-6)\left(-\frac{3\pi}{4}\right)\right)+i\sin\left((-6)\left(-\frac{3\pi}{4}\right)\right)\right)$
$=2^{6}\left(\cos\left(\frac{9\pi}{2}\right)+i\sin\left(\frac{9\pi}{2}\right)\right)$
$=2^{6}i=64i$

59. First convert $1-i$ to polar form:
$r=\sqrt{1^2+(-1)^2}=\sqrt{2};\theta=\tan^{-1}(-1)=315°$
(Note that $1-i$ lies in Quadrant III.) So,
$(1-i)^{12}=\left[\sqrt{2}(\cos 315°+i\sin 315°)\right]^{12}$
$=\left(\sqrt{2}\right)^{12}(\cos(12\cdot 315°)+i\sin(12\cdot 315°))$
$=2^{6}(\cos°3780+i\sin 3780°)=-64$

60. First convert $1-\sqrt{3}i$ to polar form:
$r=\sqrt{1^2+\left(-\sqrt{3}\right)^2}=2;\theta=\tan^{-1}\left(-\sqrt{3}\right)=300°$
(Note that $1-\sqrt{3}i$ lies in Quadrant III.)
$\left(1-\sqrt{3}i\right)^{10}=[2(\cos 300°+i\sin 300°)]^{10}$
$=2^{10}(\cos(10\cdot 300°)+i(\sin 10\cdot 300°))$
$=2^{10}(\cos 3000°+i\sin 3000°)$
$=2^{10}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\right)=-512+512i\sqrt{3}$

61. $i^{25}=i$

62. First write $3+3i$ in polar form:
$r=\sqrt{3^2+3^2}=3\sqrt{2};\theta=\tan^{-1}1=45°$ (Note that $3+3i$ lies in Quadrant I.) So
$(3+3i)^8=\left(3\sqrt{2}(\cos 45°+i\sin 45°)\right)^8$
$=\left(3\sqrt{2}\right)^8\left(\cos(8\cdot 45°)+i\sin(8\cdot 45°)\right)$
$=3^8\cdot 2^4(\cos 360°+i\sin 360°)$
$=104,976$

63. First write $\frac{1}{2}-\frac{\sqrt{3}}{2}i$ in polar form:

$$r=\sqrt{\left(\frac{1}{2}\right)^2+\left(-\frac{\sqrt{3}}{2}\right)^2}=1$$

$$\theta=\tan^{-1}\left(-\sqrt{3}\right)=300°$$

(Note that $\frac{1}{2}-\frac{\sqrt{3}}{2}i$ lies in Quadrant III.) So

$$\begin{aligned}\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^{-8} &= \left(\cos 300° + i\sin 300°\right)^{-8}\\ &= \cos(-8\cdot 300°)+i\sin(-8\cdot 300°)\\ &= \cos(-2400°)+i\sin(-2400°)\\ &= -\frac{1}{2}+\frac{\sqrt{3}}{2}i\end{aligned}$$

64. First write $\frac{\sqrt{3}}{2}+\frac{1}{2}i$ in polar form:

$$r=\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2+\left(\frac{1}{2}\right)^2}=1; \theta=\tan^{-1}\frac{\sqrt{3}}{3}=30°$$

(Note that $\frac{\sqrt{3}}{2}+\frac{1}{2}i$ lies in Quadrant I.) So

$$\begin{aligned}\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)^{10} &= \left(\cos 30° + i\sin 30°\right)^{10}\\ &= \cos(10\cdot 30°)+i\sin(10\cdot 30°)\\ &= \cos(300°)+i\sin(300°)\\ &= \frac{1}{2}-\frac{\sqrt{3}}{2}i\end{aligned}$$

65. The polar form for 64 is

$$64+0i=64(\cos 0°+i\sin 0°)$$

$$z_k=64^{1/3}\left[\cos\frac{0°+360°\cdot k}{3}+i\sin\frac{0°+360°\cdot k}{3}\right]$$

for $k = 0, 1, 2$.

$$\begin{aligned}z_0&=64^{1/3}\left(\cos\frac{0°}{3}+i\sin\frac{0°}{3}\right)\\ &=4(\cos 0°+i\sin 0°)\\ z_1&=4\left(\cos\frac{360°}{3}+i\sin\frac{360°}{3}\right)\\ &=4(\cos 120°+i\sin 120°)\\ z_2&=4\left(\cos\frac{720°}{3}+i\sin\frac{720°}{3}\right)\\ &=4(\cos 240°+i\sin 240°)\end{aligned}$$

66. The polar form for −64 is

$$-64+0i=64(\cos 180°+i\sin 180°)$$

$$z_k=64^{1/3}\left[\cos\frac{180°+360°\cdot k}{3}+i\sin\frac{180°+360°\cdot k}{3}\right]$$

for $k = 0, 1, 2$.

$$\begin{aligned}z_0&=64^{1/3}\left(\cos\frac{180°}{3}+i\sin\frac{180°}{3}\right)\\ &=4(\cos 60°+i\sin 60°)\\ z_1&=4\left(\cos\frac{540°}{3}+i\sin\frac{540°}{3}\right)\\ &=4(\cos 180°+i\sin 180°)\\ z_2&=4\left(\cos\frac{900°}{3}+i\sin\frac{900°}{3}\right)\\ &=4(\cos 300°+i\sin 300°)\end{aligned}$$

67. The polar form for i is

$$0+i=\cos 90°+i\sin 90°$$

$$z_k=\cos\frac{90°+360°\cdot k}{2}+i\sin\frac{90°+360°\cdot k}{2}$$

for $k = 0, 1$.

$$z_0=\cos\frac{90°}{2}+i\sin\frac{90°}{2}=\cos 45°+i\sin 45°$$

$$z_1=\cos\frac{450°}{2}+i\sin\frac{450°}{2}=\cos 225°+i\sin 225°$$

68. The polar form for $-i$ is

$$0-i=\cos 270°+i\sin 270°$$

$$z_k=\cos\frac{270°+360°\cdot k}{4}+i\sin\frac{270°+360°\cdot k}{4}$$

for $k = 0, 1, 2, 3$.

$$\begin{aligned}z_0&=\cos\frac{270°}{4}+i\sin\frac{270°}{4}\\ &=\cos 67.5°+i\sin 67.5°\\ z_1&=\cos\frac{630°}{4}+i\sin\frac{630°}{4}\\ &=\cos 157.5°+i\sin 157.5°\\ z_2&=\cos\frac{990°}{4}+i\sin\frac{990°}{4}\\ &=\cos 247.5°+i\sin 247.5°\\ z_3&=\cos\frac{1350°}{4}+i\sin\frac{1350°}{4}\\ &=\cos 337.5°+i\sin 337.5°\end{aligned}$$

69. The polar form for -1 is

$-1+0i=\cos 180°+i\sin 180°$

$$z_k=\cos\frac{180°+360°\cdot k}{6}+i\sin\frac{180°+360°\cdot k}{6}$$

for $k=0, 1, 2, 3, 4, 5.$

$$z_0=\cos\frac{180°}{6}+i\sin\frac{180°}{6}=\cos 30°+i\sin 30°$$

$$z_1=\cos\frac{540°}{6}+i\sin\frac{540°}{6}=\cos 90°+i\sin 90°$$

$$z_2=\cos\frac{900°}{6}+i\sin\frac{900°}{6}=\cos 150°+i\sin 150°$$

$$z_3=\cos\frac{1260°}{6}+i\sin\frac{1260°}{6}=\cos 210°+i\sin 210°$$

$$z_4=\cos\frac{1620°}{6}+i\sin\frac{1620°}{6}=\cos 270°+i\sin 270°$$

$$z_5=\cos\frac{1980°}{6}+i\sin\frac{1980°}{6}=\cos 330°+i\sin 330°$$

70. The polar form for 1 is

$1+0i=\cos 0°+i\sin 0°$

$$z_k=\cos\frac{0°+360°\cdot k}{8}+i\sin\frac{0°+360°\cdot k}{8}$$ for

$k=0, 1, 2, 3, 4, 5, 6, 7.$

$$z_0=\cos\frac{0°}{8}+i\sin\frac{0°}{8}=\cos 0°+i\sin 0°$$

$$z_1=\cos\frac{360°}{8}+i\sin\frac{360°}{8}=\cos 45°+i\sin 45°$$

$$z_2=\cos\frac{720°}{8}+i\sin\frac{720°}{8}=\cos 90°+i\sin 90°$$

$$z_3=\cos\frac{1080°}{8}+i\sin\frac{1080°}{8}=\cos 135°+i\sin 135°$$

$$z_4=\cos\frac{1440°}{8}+i\sin\frac{1440°}{8}=\cos 180°+i\sin 180°$$

$$z_5=\cos\frac{1800°}{8}+i\sin\frac{1800°}{8}=\cos 225°+i\sin 225°$$

$$z_6=\cos\frac{2160°}{8}+i\sin\frac{2160°}{8}=\cos 270°+i\sin 270°$$

$$z_7=\cos\frac{2520°}{8}+i\sin\frac{2520°}{8}=\cos 315°+i\sin 315°$$

71. Write $1-\sqrt{3}i$ in polar form:

$$r=\sqrt{1^2+\left(-\sqrt{3}\right)^2}=2;\theta=\tan^{-1}\left(-\sqrt{3}\right)=300°$$

(Note that $1-\sqrt{3}i$ lies in Quadrant III.)

$$z_k=2^{1/2}\left(\cos\frac{300°+360°\cdot k}{2}+i\sin\frac{300°+360°\cdot k}{2}\right)$$

for $k=0, 1.$

$$z_0=\sqrt{2}\left(\cos\frac{300°}{2}+i\sin\frac{300°}{2}\right)=\sqrt{2}\left(\cos 150°+i\sin 150°\right)$$

$$z_1=\sqrt{2}\left(\cos\frac{660°}{2}+i\sin\frac{660°}{2}\right)=\sqrt{2}\left(\cos 330°+i\sin 330°\right)$$

72. Write $4+4\sqrt{3}i$ in polar form:

$$r=\sqrt{4^2+\left(4\sqrt{3}\right)^2}=8;\theta=\tan^{-1}\left(\sqrt{3}\right)=60°$$

(Note that $4+4\sqrt{3}i$ lies in Quadrant I.)

$$z_k=8^{1/3}\left(\cos\frac{60°+360°\cdot k}{3}+i\sin\frac{60°+360°\cdot k}{3}\right)$$

for $k=0, 1, 2.$

$$z_0=8^{1/3}\left(\cos\frac{60°}{3}+i\sin\frac{60°}{3}\right)=2\left(\cos 20°+i\sin 20°\right)$$

$$z_1=8^{1/3}\left(\cos\frac{420°}{3}+i\sin\frac{420°}{3}\right)=2\left(\cos 140°+i\sin 140°\right)$$

$$z_2=8^{1/3}\left(\cos\frac{780°}{3}+i\sin\frac{780°}{3}\right)=2\left(\cos 260°+i\sin 260°\right)$$

7.7 Applying the Concepts

73. By DeMoivre's theorem,

$\cos 3\theta+i\sin 3\theta=(\cos\theta+i\sin\theta)^3.$ Expanding, we have

$$(\cos\theta+i\sin\theta)^3=\cos^3\theta+3\cos^2\theta i\sin\theta-3\cos\theta\sin^2\theta-i\sin^3\theta$$

a. To solve for $\sin 3\theta$, gather the even-powered cosine terms and simplify:

$$i\sin 3\theta=3i\cos^2\theta\sin\theta-i\sin^3\theta=i\sin\theta(3\cos^2\theta-\sin^2\theta)\Rightarrow$$

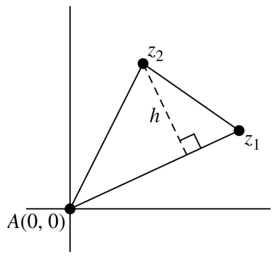
$$\sin 3\theta=\sin\theta(3\cos^2\theta-\sin^2\theta)=\sin\theta(3(1-\sin^2\theta)-\sin^2\theta)=\sin\theta(3-4\sin^2\theta)=3\sin\theta-4\sin^3\theta$$

b. To solve for $\cos 3\theta$, gather the odd-powered cosine terms and simplify:

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta = \cos\theta(\cos^2\theta - 3\sin^2\theta) \Rightarrow$$

$$\cos 3\theta = \cos\theta(\cos^2\theta - 3\sin^2\theta) = \cos\theta(\cos^2\theta - 3(1-\cos^2\theta)) = \cos\theta(4\cos^2\theta - 3) = 4\cos^3\theta - 3\cos\theta$$

74.



$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$m\angle A = |\theta_2 - \theta_1| = \left|\arg\frac{z_1}{z_2}\right|.$

$|z_1| = |r_1|$ and $|z_2| = |r_2|$, so

$h = |z_2|\sin\left(\left|\arg\frac{z_1}{z_2}\right|\right)$. Therefore,

$$K = \frac{1}{2}|z_1||z_2|\sin\left(\left|\arg\frac{z_1}{z_2}\right|\right)$$

75. $I = \frac{V}{Z} = \frac{120(\cos 60° + i\sin 60°)}{8(\cos 30° + i\sin 30°)} = 15(\cos 30° + i\sin 30°)$

76. $V = IZ = 6(\cos 40° + i\sin 40°) \cdot 16(\cos 110° + i\sin 110°) = 96(\cos 150° + i\sin 150°)$

77. To find $z_1 + z_2$, first convert each to rectangular form. Then convert the sum back to polar form:

$z_1 = 16(\cos 180° + i\sin 180°) = -16$

$z_2 = 2(\cos 150° + i\sin 150°) = -\sqrt{3} + i$

$z_1 + z_2 = -16 - \sqrt{3} + i$

$r = \sqrt{\left(-16-\sqrt{3}\right)^2 + 1^2} \approx 17.76$

$\theta = \tan^{-1}\left(\frac{1}{-16-\sqrt{3}}\right) \approx -3.2° = 176.8°$. So

$z_1 + z_2 \approx 17.76(\cos(176.8°) + i\sin(176.8°))$

$z_1z_2 = 32(\cos 330° + i\sin 330°)$

$$\frac{z_1z_2}{z_1+z_2} \approx \frac{32(\cos 330° + i\sin 330°)}{17.76(\cos(176.8°) + i\sin(176.8°))} \approx 1.8(\cos 153.2° + i\sin 153.2°)$$

78. To find $z_1 + z_2$, first convert each to rectangular form. Then convert the sum back to polar form:

$z_1 = 12(\cos 270° + i\sin 270°) = -12i$

$z_2 = 3(\cos 60° + i\sin 60°) = \frac{3}{2} + \frac{3\sqrt{3}}{2}i$

$z_1 + z_2 = \frac{3}{2} + \left(\frac{3\sqrt{3}}{2} - 12\right)i$

$r = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{3\sqrt{3}}{2} - 12\right)^2} \approx 9.52$

$\theta = \tan^{-1}\left(\frac{\frac{3\sqrt{3}}{2} - 12}{\frac{3}{2}}\right) \approx -80.9° = 99.1°$. So

$z_1 + z_2 \approx 9.52(\cos 99.1° + i\sin 99.1°)$

$z_1z_2 = 36(\cos 330° + i\sin 330°)$

$$\frac{z_1z_2}{z_1+z_2} \approx \frac{36(\cos 330° + i\sin 330°)}{9.52(\cos 99.1° + i\sin 99.1°)} \approx 3.8(\cos 230.9° + i\sin 230.9°) \approx 3.8(\cos 50.9° + i\sin 50.9°)$$

7.7 Beyond the Basics

79. First convert each factor to polar form:

$z_1 = 1 - \sqrt{3}i = 2(\cos 300° + i\sin 300°)$

$z_2 = -2 + 2i = 2\sqrt{2}(\cos 135° + i\sin 135°)$

Now use DeMoivre's theorem to raise each factor to the indicated power:

$$z_1^{10} = \left(1 - \sqrt{3}i\right)^{10} = [2(\cos 300° + i\sin 300°)]^{10} = 2^{10}(\cos 3000° + i\sin 3000°)$$

$$z_2^{-6} = (-2+2i)^{-6} = \left[2\sqrt{2}(\cos 135° + i\sin 135°)\right]^{-6} = 2^{-9}(\cos(-810°) + i\sin(-810°))$$

Multiply the results:

$$z_1^{10}z_2^{-6} = 2^{10}(\cos 3000° + i\sin 3000°) \cdot 2^{-9}(\cos(-810°) + i\sin(-810°)) = 2(\cos 2190° + i\sin 2190°) = 2(\cos 30° + i\sin 30°)$$

80. First convert each factor to polar form:

$$z_1 = \frac{1}{2} + \frac{\sqrt{3}}{2}i = \cos 60° + i\sin 60°$$
$$z_2 = 2 - 2i = 2\sqrt{2}(\cos(-45°) + i\sin(-45°))$$

Now use DeMoivre's theorem to raise each factor to the indicated power:

$$z_1{}^8 = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^8 = \left[\cos 60° + i\sin 60°\right]^8 = \cos 480° + i\sin 480°$$

$$z_2{}^{-6} = (2-2i)^{-6} = \left[2\sqrt{2}(\cos(-45°) + i\sin(-45°))\right]^{-6} = 2^{-9}(\cos 270° + i\sin 270°)$$

Multiply the results:

$$z_1{}^8 z_2{}^{-6} = (\cos 480° + i\sin 480°)\cdot 2^{-9}(\cos(270°) + i\sin(270°)) = \frac{1}{512}(\cos 750° + i\sin 750°)$$
$$= \frac{1}{512}(\cos 30° + i\sin 30°)$$

81. $$\left(\sin\frac{\pi}{6} + i\cos\frac{\pi}{6}\right)^{10} = (\sin 30° + i\cos 30°)^{10} = (\cos 60° + i\sin 60°)^{10} = \cos 600° + i\sin 600°$$
$$= \cos 240° + i\sin 240°$$

82. $$\left(\sin\frac{2\pi}{3} + i\cos\frac{2\pi}{3}\right)^{6} = (\sin 120° + i\cos 120°)^{6} = (\cos 330° + i\sin 330°)^{6} = \cos 1980° + i\sin 1980°$$
$$= \cos 180° + i\sin 180°$$

83. $$\left(\sin\frac{5\pi}{3} + i\cos\frac{5\pi}{3}\right)^{-8} = (\sin 300° + i\cos 300°)^{-8} = (\cos 150° + i\sin 150°)^{-8}$$
$$= \cos(-1200°) + i\sin(-1200°) = \cos 240° + i\sin 240°$$

84. $$\left(\sin\frac{7\pi}{6} + i\cos\frac{7\pi}{6}\right)^{-10} = (\sin 210° + i\cos 210°)^{-10} = (\cos 240° + i\sin 240°)^{-10}$$
$$= \cos(-2400°) + i\sin(-2400°) = \cos 120° + i\sin 120°$$

85. $z^4 = 1 \Rightarrow z^4 = \cos 0° + i\sin 0° \Rightarrow z = (\cos 0° + i\sin 0°)^{1/4}$

$$z_k = \cos\frac{0° + 360°\cdot k}{4} + i\sin\frac{0° + 360°\cdot k}{4}, \text{ for } k = 0, 1, 2, 3.$$
$$z_0 = \cos\frac{0°}{4} + i\sin\frac{0°}{4} = \cos 0° + i\sin 0°$$
$$z_1 = \cos\frac{360°}{4} + i\sin\frac{360°}{4} = \cos 90° + i\sin 90°$$
$$z_2 = \cos\frac{720°}{4} + i\sin\frac{720°}{4} = \cos 180° + i\sin 180°$$
$$z_3 = \cos\frac{1080°}{4} + i\sin\frac{1080°}{4} = \cos 270° + i\sin 270°$$

86. $z^8 = -1 \Rightarrow z^8 = \cos 180° + i\sin 180° \Rightarrow z = (\cos 180° + i\sin 180°)^{1/8}$

$$z_k = \cos\frac{0° + 360°\cdot k}{8} + i\sin\frac{0° + 360°\cdot k}{8}, \text{ for } k = 0, 1, 2, 3, 4, 5, 6, 7.$$
$$z_0 = \cos\frac{180°}{8} + i\sin\frac{180°}{8} = \cos 22.5° + i\sin 22.5°$$
$$z_1 = \cos\frac{540°}{8} + i\sin\frac{540°}{8} = \cos 67.5° + i\sin 67.5°$$

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$$z_2 = \cos\frac{900°}{8} + i\sin\frac{900°}{8} = \cos 112.5° + i\sin 112.5°$$
$$z_3 = \cos\frac{1260°}{8} + i\sin\frac{1260°}{8} = \cos 157.5° + i\sin 157.5°$$
$$z_4 = \cos\frac{1620°}{8} + i\sin\frac{1620°}{8} = \cos 202.5° + i\sin 202.5°$$
$$z_5 = \cos\frac{1980°}{8} + i\sin\frac{1980°}{8} = \cos 247.5° + i\sin 247.5°$$
$$z_6 = \cos\frac{2340°}{8} + i\sin\frac{2340°}{8} = \cos 292.5° + i\sin 292.5°$$
$$z_7 = \cos\frac{2700°}{8} + i\sin\frac{2700°}{8} = \cos 337.5° + i\sin 337.5°$$

87. $z^3 = 1+i \Rightarrow z^3 = \sqrt{2}(\cos 45° + i\sin 45°) \Rightarrow z = \left(\sqrt{2}(\cos 45° + i\sin 45°)\right)^{1/3}$

$$z_k = 2^{1/6}\left(\cos\frac{45° + 360°\cdot k}{3} + i\sin\frac{45° + 360°\cdot k}{3}\right), \text{ for } k = 0, 1, 2.$$
$$z_0 = 2^{1/6}\left(\cos\frac{45°}{3} + i\sin\frac{45°}{3}\right) = 2^{1/6}\left(\cos 15° + i\sin 15°\right)$$
$$z_1 = 2^{1/6}\left(\cos\frac{405°}{3} + i\sin\frac{405°}{3}\right) = 2^{1/6}\left(\cos 135° + i\sin 135°\right)$$
$$z_2 = 2^{1/6}\left(\cos\frac{765°}{3} + i\sin\frac{765°}{3}\right) = 2^{1/6}\left(\cos 255° + i\sin 255°\right)$$

88. $z^7 = (1-i)^2 \Rightarrow z^7 = \left[\sqrt{2}(\cos(-45°) + i\sin(-45°))\right]^2 \Rightarrow z^7 = \left(2(\cos 270° + i\sin 270°)\right) \Rightarrow$

$z = \left(2(\cos 270° + i\sin 270°)\right)^{1/7} \Rightarrow z_k = 2^{1/7}\left(\cos\frac{270° + 360°\cdot k}{7} + i\sin\frac{270° + 360°\cdot k}{7}\right)$

for $k = 0, 1, 2, 3, 4, 5, 6.$

$$z_0 = 2^{1/7}\left(\cos\frac{270°}{7} + i\sin\frac{270°}{7}\right) = 2^{1/7}\left(\cos 38.6° + i\sin 38.6°\right)$$
$$z_1 = 2^{1/7}\left(\cos\frac{630°}{7} + i\sin\frac{630°}{7}\right) = 2^{1/7}\left(\cos 90° + i\sin 90°\right)$$
$$z_2 = 2^{1/7}\left(\cos\frac{990°}{7} + i\sin\frac{990°}{7}\right) = 2^{1/7}\left(\cos 141.4° + i\sin 141.4°\right)$$
$$z_3 = 2^{1/7}\left(\cos\frac{1350°}{7} + i\sin\frac{1350°}{7}\right) = 2^{1/7}\left(\cos 192.9° + i\sin 192.9°\right)$$
$$z_4 = 2^{1/7}\left(\cos\frac{1710°}{7} + i\sin\frac{1710°}{7}\right) = 2^{1/7}\left(\cos 244.3° + i\sin 244.3°\right)$$
$$z_5 = 2^{1/7}\left(\cos\frac{2070°}{7} + i\sin\frac{2070°}{7}\right) = 2^{1/7}\left(\cos 295.7° + i\sin 295.7°\right)$$
$$z_6 = 2^{1/7}\left(\cos\frac{2430°}{7} + i\sin\frac{2430°}{7}\right) = 2^{1/7}\left(\cos 347.1° + i\sin 347.1°\right)$$

89. If $z = \cos\theta + i\sin\theta, z \neq 0$, then $\frac{1}{z} = z^{-1} = (\cos\theta + i\sin\theta)^{-1} = \cos(-\theta) + i\sin(-\theta)$

$\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$, so this becomes $\cos\theta - i\sin\theta$.

90. $z+\dfrac{1}{z}=(\cos\theta+i\sin\theta)+(\cos\theta-i\sin\theta)=2\cos\theta$

91. $z-\dfrac{1}{z}=(\cos\theta+i\sin\theta)-(\cos\theta-i\sin\theta)=2i\sin\theta$

92. $$z^n+\frac{1}{z^n}=(\cos\theta+i\sin\theta)^n+(\cos\theta+i\sin\theta)^{-n}=(\cos n\theta+i\sin n\theta)+(\cos(-n\theta)+i\sin(-n\theta))$$
$$=(\cos n\theta+i\sin n\theta)+(\cos n\theta-i\sin n\theta)=2\cos n\theta$$

93. $$z^n-\frac{1}{z^n}=(\cos\theta+i\sin\theta)^n-(\cos\theta+i\sin\theta)^{-n}=(\cos n\theta+i\sin n\theta)-(\cos(-n\theta)+i\sin(-n\theta))$$
$$=(\cos n\theta+i\sin n\theta)-(\cos n\theta-i\sin n\theta)=2i\sin n\theta$$

94. Using the double-angle formulas, $(1+\cos\theta+i\sin\theta)^n=\left(2\cos^2\dfrac{\theta}{2}+2i\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}\right)^n$.

Factor: $\left(2\cos^2\dfrac{\theta}{2}+2i\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}\right)^n=\left(2\cos\dfrac{\theta}{2}\left(\cos\dfrac{\theta}{2}+2i\sin\dfrac{\theta}{2}\right)\right)^n$.

Use DeMoivre's theorem: $\left(2\cos\dfrac{\theta}{2}\left(\cos\dfrac{\theta}{2}+2i\sin\dfrac{\theta}{2}\right)\right)^n=\left(2\cos\dfrac{\theta}{2}\right)^n\left(\cos\dfrac{n\theta}{2}+2i\sin\dfrac{n\theta}{2}\right)$

95. Let $z_1=r_1(\cos\theta_1+i\sin\theta_1)$ and $z_2=r_2(\cos\theta_2+i\sin\theta_2)$. Then
$$\begin{aligned}z_1z_2&=r_1(\cos\theta_1+i\sin\theta_1)\cdot r_2(\cos\theta_2+i\sin\theta_2)\\&=r_1r_2(\cos\theta_1\cos\theta_2+i\cos\theta_1\sin\theta_2+i\sin\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2)\\&=r_1r_2\left[(\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2)+i(\cos\theta_1\sin\theta_2+\sin\theta_1\cos\theta_2)\right]\\&=r_1r_2\left[\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)\right]\end{aligned}$$

96. Let $z_1=r_1(\cos\theta_1+i\sin\theta_1)$ and $z_2=r_2(\cos\theta_2+i\sin\theta_2)$. Then
$$\begin{aligned}\frac{z_1}{z_2}&=\frac{r_1(\cos\theta_1+i\sin\theta_1)}{r_2(\cos\theta_2+i\sin\theta_2)}=\frac{r_1(\cos\theta_1+i\sin\theta_1)}{r_2(\cos\theta_2+i\sin\theta_2)}\cdot\frac{(\cos\theta_2-i\sin\theta_2)}{(\cos\theta_2-i\sin\theta_2)}\\&=\frac{r_1(\cos\theta_1\cos\theta_2-i\cos\theta_1\sin\theta_2+i\sin\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2)}{r_2(\cos^2\theta_2+\sin^2\theta_2)}\\&=\frac{r_1}{r_2}\cdot\left[(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2)+i(\sin\theta_1\cos\theta_2-\cos\theta_1\sin\theta_2)\right]=\frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)\right]\end{aligned}$$

97. $(3+2i)(5+i)=15+13i-2=13+13i$

We want to prove that $\tan^{-1}\left(\dfrac{2}{3}\right)+\tan^{-1}\left(\dfrac{1}{5}\right)=\dfrac{\pi}{4}$. Since complex multiplication adds arguments, multiply the complex numbers with arguments of $\tan^{-1}\left(\dfrac{2}{3}\right)$ and $\tan^{-1}\left(\dfrac{1}{5}\right)$, namely, $(3+2i)(5+i)$. The product has an argument of $\tan^{-1}\left(\dfrac{13}{13}\right)=\tan^{-1}(1)=\dfrac{\pi}{4}$.

98. The argument of $((p+q)+i)$ is $\tan^{-1}\left(\frac{1}{p+q}\right)$ and the argument of $\left((p^2+pq+1)+iq\right)$ is $\tan^{-1}\left(\frac{q}{p^2+pq+1}\right)$. Since complex multiplication adds arguments, multiply the complex numbers to find the sum of the arguments.

$$\begin{aligned}((p+q)+i)\left((p^2+pq+1)+iq\right) &= (p+q)\left[(p^2+pq+1)+iq\right]+i\left[(p^2+pq+1)+iq\right]\\ &= p^3+p^2q+p^2q+pq^2+p+q+ipq+iq^2+ip^2+ipq+i-q\\ &= p^3+2p^2q+pq^2+p+iq^2+2ipq+ip^2+i\\ &= \left(p^3+2p^2q+pq^2+p\right)+i\left((p^2+2pq+q^2)+1\right)\end{aligned}$$

The argument of the product is $\tan^{-1}\left(\frac{(p^2+2pq+q^2)+1}{p^3+2p^2q+pq^2+p}\right)=\tan^{-1}\left(\frac{(p+q)^2+1}{p\left((p+q)^2+1\right)}\right)=\tan^{-1}\frac{1}{p}$.

Thus, $\tan^{-1}\left(\frac{1}{p+q}\right)+\tan^{-1}\left(\frac{q}{p^2+pq+1}\right)=\tan^{-1}\left(\frac{1}{p}\right)$.

99. DeMoivre's theorem states that for any integer n, $z^n=r^n(\cos n\theta+i\sin n\theta)$, so the argument is $n\tan^{-1}\theta$.

Thus, the argument of $(2+3i)^4$ is $4\tan^{-1}\left(\frac{3}{2}\right)$.

$$(2+3i)^4=\left((2+3i)^2\right)^2=(4+12i-9)^2=(-5+12i)^2=25-120i-144=-119-120i$$

The argument of $(-119-120i)$ is $\tan^{-1}\left(\frac{120}{119}\right)$. Since complex division subtracts arguments, divide the complex numbers to find the difference of the arguments.

$$4\tan^{-1}\left(\frac{3}{2}\right)-\tan^{-1}\left(\frac{120}{119}\right)=\arg\left(\frac{(2+3i)^4}{-119-120i}\right)=\arg(-1+0i)=\tan^{-1}(0)=\pi$$

100. DeMoivre's theorem states that for any integer n, $z^n=r^n(\cos n\theta+i\sin n\theta)$, so the argument is $n\tan^{-1}\theta$.

Thus, the argument of $(5+i)^4$ is $4\tan^{-1}\left(\frac{1}{5}\right)$.

The argument of $(-239+i)$ is $\tan^{-1}\left(\frac{1}{-239}\right)=-\tan^{-1}\left(\frac{1}{239}\right)$.

$$(5+i)^4(-239+i)=(476+480i)(-239+i)=-114,244-114,244i$$

$$4\tan^{-1}\left(\frac{1}{5}\right)-\tan^{-1}\left(\frac{1}{239}\right)=\arg\left((5+i)^4(-239+i)\right)=\arg(-114,244-114,244i)=\tan^{-1}(1)=\frac{\pi}{4}$$

101. $x^3+x^2+x+1=0\Rightarrow(x-1)\left(x^3+x^2+x+1\right)=0\Rightarrow x^4-1=0\Rightarrow\left(x^2+1\right)\left(x^2-1\right)=0\Rightarrow x=\pm i,\ \pm 1$

We reject the root $x = 1$ since it was introduced when the equation was multiplied by $x - 1$.
Solution set: $\{-i, i, -1\}$

102. $x^4+x^3+x^2+x+1=0\Rightarrow(x-1)\left(x^4+x^3+x^2+x+1\right)=0\Rightarrow x^5-1=0\Rightarrow x^5=1$

$z^5=1\Rightarrow z^5=\cos 0°+i\sin 0°\Rightarrow z=(\cos 0°+i\sin 0°)^{1/5}$

$z_k=\cos\frac{0°+360°\cdot k}{5}+i\sin\frac{0°+360°\cdot k}{5}$, for $k=$ 0, 1, 2, 3, 4.

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$z_0 = \cos\frac{0°}{5} + i\sin\frac{0°}{5} = \cos 0° + i\sin 0° = 1$ $\quad z_1 = \cos\frac{360°}{5} + i\sin\frac{360°}{5} = \cos 72° + i\sin 72°$

$z_2 = \cos\frac{720°}{5} + i\sin\frac{720°}{5} = \cos 144° + i\sin 144°$ $\quad z_3 = \cos\frac{1080°}{5} + i\sin\frac{1080°}{5} = \cos 216° + i\sin 216°$

$z_4 = \cos\frac{1440°}{5} + i\sin\frac{1440°}{5} = \cos 288° + i\sin 288°$

We reject the root $x = 1$ since it was introduced when the equation was multiplied by $x - 1$.

Solution set: $\{\cos 72° + i\sin 72°,\ \cos 144° + i\sin 144°,\ \cos 216° + i\sin 216°,\ \cos 288° + i\sin 288°\}$

7.7 Critical Thinking/Discussion/Writing

103. The exercises that are false are incorrect usages of DeMoivre's theorem.

a. True **b.** False

c. True **d.** False

e. True **f.** True

g. True **h.** True

i. False **j.** True

7.7 Maintaining Skills

104. $3x + 2y = 7, \quad y = \frac{1}{2}$

Substituting, we have

$3x + 2\left(\frac{1}{2}\right) = 7 \Rightarrow 3x + 1 = 7 \Rightarrow 3x = 6 \Rightarrow$
$x = 2$

105. $-2x + 9y = 5, \quad y = 3$

Substituting, we have

$-2x + 9(3) = 5 \Rightarrow -2x + 27 = 5 \Rightarrow$
$-2x = -22 \Rightarrow x = 11$

106. $-4x + 7y = 7, \quad x = 0$

Substituting, we have

$-4(0) + 7y = 7 \Rightarrow 7y = 7 \Rightarrow y = 1$

107. $23x - 14y = -5, \quad x = 1$

Substituting, we have

$23(1) - 14y = -5 \Rightarrow 23 - 14y = -5 \Rightarrow$
$-14y = -28 \Rightarrow y = 2$

108. Rearrange the equation to put it in slope-intercept form.

$10x - 2y = 28 \Rightarrow -2y = -10x + 28 \Rightarrow$
$y = 5x - 14$

The slope is 5.

109. Rearrange the equation to put it in slope-intercept form.

$15x + 5y = 2 \Rightarrow 5y = -15x + 2 \Rightarrow y = -3x + \frac{2}{5}$

The slope is −3.

110. Parallel lines have the same slope. Rearrange the equation to put it in slope-intercept form.

$6x - 3y = 7 \Rightarrow -3y = -6x + 7 \Rightarrow$
$y = 3x - \frac{7}{3}$

The slope is 2. Now use the point-slope form to write the equation of the parallel line.

$y - (-1) = 2(x - 5) \Rightarrow y + 1 = 2(x - 5) \Rightarrow$
$y + 1 = 2x - 10 \Rightarrow 2x - y = 11$

111. Parallel lines have the same slope. Rearrange the equation to put it in slope-intercept form.

$x + 2y = 12 \Rightarrow 2y = -x + 12 \Rightarrow y = -\frac{1}{2}x + 6$

The slope is $-\frac{1}{2}$. Now use the point-slope form to write the equation of the parallel line.

$y - 3 = -\frac{1}{2}(x - 3) \Rightarrow 2y - 6 = -x + 3 \Rightarrow$
$2y + x = 9$

112. Two equations have the same graph if the coefficients of one of the equations are multiples are the coefficients of the second equation. So any equation that has the same graph as the equation $2x - 4y = 12$ will have the form $2nx - 4ny = 12n$. We are given that $12n = 3$, so $n = \frac{1}{4}$. Therefore, $a = 2 \cdot \frac{1}{4} = \frac{1}{2}$ and $b = -4 \cdot \frac{1}{4} = -1$.

113. Two equations have the same graph if the coefficients of one of the equations are multiples are the coefficients of the second equation. So any equation that has the same graph as the equation $4x - y = -1$ will have the form $4nx - ny = -n$. We are given that $-n = 2$, so $n = -2$ Therefore, $a = 4(-2) = -8$ and $b = -1(-2) = 2$.

Chapter 7 Review Exercises

Basic Concepts and Skills

1. $B = 90° - 30° = 60°$
$\sin A = \frac{a}{c} \Rightarrow \sin 30° = \frac{6}{c} \Rightarrow c = 12$
$b = \sqrt{c^2 - a^2} = \sqrt{12^2 - 6^2} = \sqrt{108} = 6\sqrt{3} \approx 10.4$

2. $A = 90° - 35° = 55°$
$\sin B = \frac{b}{c} \Rightarrow \sin 35° = \frac{5}{c} \Rightarrow c \approx 8.7$
$a = \sqrt{c^2 - b^2} \approx \sqrt{8.7^2 - 5^2} \approx 7.1$

3. $B = 90° - 37° = 53°$
$\sin B = \frac{b}{c} \Rightarrow \sin 53° = \frac{4}{c} \Rightarrow c = 5.0$
$a = \sqrt{c^2 - b^2} = \sqrt{5.0^2 - 4^2} \approx 3.0$

4. $A = 90° - 43° = 47°$
$\sin A = \frac{a}{c} \Rightarrow \sin 47° = \frac{10}{c} \Rightarrow c \approx 13.7$
$b = \sqrt{c^2 - a^2} \approx \sqrt{13.7^2 - 10^2} \approx 9.4$

5. $B = 90° - 40° = 50°$
$\sin A = \frac{a}{c} \Rightarrow \sin 40° = \frac{a}{12} \Rightarrow a \approx 7.7$
$b = \sqrt{c^2 - a^2} = \sqrt{5.0^2 - 4^2} \approx 9.2$

6. $A = 90° - 50° = 40°$
$\sin B = \frac{b}{c} \Rightarrow \sin 50° = \frac{b}{8} \Rightarrow b \approx 6.1$
$a = \sqrt{c^2 - b^2} \approx \sqrt{8^2 - (8\sin 50°)^2} \approx 5.1$

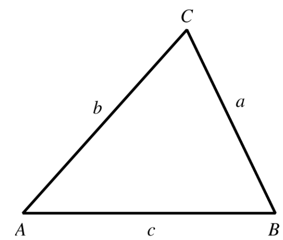
7. $c = \sqrt{a^2 + b^2} = \sqrt{3^2 + 5^2} = \sqrt{34} \approx 5.8$
$A = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{5}{3}\right) \approx 59.0°$
$B \approx 90° - 59.0° = 31.0°$

8. $c = \sqrt{a^2 + b^2} = \sqrt{5^2 + 10^2} = \sqrt{125} \approx 11.2$
$A = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{10}{5}\right) \approx 63.4°$
$B \approx 90° - 63.4° = 26.6°$

9. $b = \sqrt{c^2 - a^2} = \sqrt{6^2 - 4^2} = \sqrt{20} \approx 4.5$
$A = \sin^{-1}\left(\frac{a}{c}\right) = \sin^{-1}\left(\frac{4}{6}\right) \approx 41.8°$
$B \approx 90° - 41.8° = 48.2°$

10. $a = \sqrt{c^2 - b^2} = \sqrt{7^2 - 3^2} = \sqrt{40} \approx 6.3$
$B = \sin^{-1}\left(\frac{b}{c}\right) = \sin^{-1}\left(\frac{3}{7}\right) \approx 25.4°$
$B \approx 90° - 25.4° = 64.6°$

Use this triangle to help solve exercises 11−18.



11. Given: $A = 40°, B = 35°, c = 100$ − an ASA case. $C = 180° - (40° + 35°) = 105°$
$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \frac{a}{\sin 40°} = \frac{100}{\sin 105°} \Rightarrow a \approx 66.5$
$\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{b}{\sin 35°} = \frac{100}{\sin 105°} \Rightarrow b \approx 59.4$

12. Given: $B = 30°, C = 80°, a = 100$ − an ASA case. $A = 180° - (30° + 80°) = 70°$
$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{100}{\sin 70°} = \frac{b}{\sin 30°} \Rightarrow b \approx 53.2$
$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \frac{100}{\sin 70°} = \frac{c}{\sin 80°} \Rightarrow c \approx 104.8$

13. Given: $A = 45°, a = 25, b = 75$ − an SSA case. A is an acute angle, so examine h:
$h = b\sin A = 75\sin 45° \approx 43.3$
$a < h$, so no triangle exists.

14. Given: $B = 36°, a = 12.5, b = 8.7$ − an SSA case. B is an acute angle, so examine h:
$h = a\sin B = 12.5\sin 36° \approx 7.3$
$h < b < a$, so two triangles exist.

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$\frac{\sin B}{b} = \frac{\sin A}{a} \Rightarrow$

$\frac{\sin 36°}{8.7} = \frac{\sin A}{12.5} \Rightarrow A = \sin^{-1}\left(\frac{12.5 \sin 36°}{87}\right) \Rightarrow$

$A_1 \approx 57.6°, A_2 \approx 122.4°$

$C_1 = 180° - (36° + 57.6°) \approx 86.4°$

$\frac{b}{\sin B} = \frac{c_1}{\sin C_1} \Rightarrow \frac{8.7}{\sin 36°} = \frac{c_1}{\sin 86.4°} \Rightarrow$

$c_1 \approx 14.8$

$C_2 = 180° - (36° + 122.4°) \approx 21.6°$

$\frac{b}{\sin B} = \frac{c_2}{\sin C_2} \Rightarrow \frac{8.7}{\sin 36°} = \frac{c_2}{\sin 21.6°} \Rightarrow$

$c_2 \approx 5.4$.

The two solutions are: $A_1 \approx 57.6°$, $C_1 \approx 86.4°, c_1 \approx 14.8$ and $A_2 \approx 122.4°$, $C_2 \approx 21.6°, c_2 \approx 5.4$.

15. Given: $A = 48.5°, C = 57.3°, b = 47.3$ – an ASA case. $B = 180° - (48.5° + 57.3°) = 74.2°$

$\frac{b}{\sin B} = \frac{a}{\sin A} \Rightarrow \frac{47.3}{\sin 74.2°} = \frac{a}{\sin 48.5°} \Rightarrow$

$a \approx 36.8$

$\frac{b}{\sin B} = \frac{c}{\sin A} \Rightarrow \frac{47.3}{\sin 74.2°} = \frac{c}{\sin 57.3°} \Rightarrow$

$c \approx 41.4$

16. Given: $A = 67°, a = 100, c = 125$ – an SSA case. A is an acute angle, so examine h:

$h = c \sin A = 125 \sin 67° \approx 115$

$a < h$ so no triangle exists.

17. Given: $A = 65.2°, a = 21.3, b = 19$ – an SSA case. A is an acute angle, so examine h:

$h = b \sin A = 19 \sin 65.2° \approx 17.2$. $h < b < a$, so there is one triangle. $\frac{\sin 65.2°}{21.3} = \frac{\sin B}{19} \Rightarrow$

$B = \sin^{-1}\left(\frac{19 \sin 65.2°}{21.3}\right) \approx 54.1°$

$C \approx 180° - (65.2° + 54.1°) \approx 60.7°$

$\frac{21.3}{\sin 65.2°} = \frac{c}{\sin 60.7°} \Rightarrow c \approx 20.5$

18. Given: $C = 53°, a = 140, c = 115$ – an SSA case. C is an acute angle, so examine h:

$h = a \sin C = 140 \sin 53° \approx 111.8$

$h < c < a$, so two triangles exist.

$\frac{\sin C}{c} = \frac{\sin A}{a} \Rightarrow \frac{\sin 53°}{115} = \frac{\sin A}{140} \Rightarrow$

$A_1 = \sin^{-1}\left(\frac{140 \sin 53°}{115}\right) \Rightarrow A_1 \approx 76.5°$,

$A_2 \approx 103.5°$

$B_1 \approx 180° - (53° + 76.5°) \approx 50.5°$

$\frac{c}{\sin C} = \frac{b_1}{\sin B_1} \Rightarrow \frac{115}{\sin 53°} = \frac{b_1}{\sin 50.5°} \Rightarrow$

$b_1 \approx 111.1$

$B_2 \approx 180° - (53° + 103.5°) \approx 23.5°$

$\frac{c}{\sin C} = \frac{b_2}{\sin B_2} \Rightarrow \frac{115}{\sin 53°} = \frac{b_2}{\sin 23.5°} \Rightarrow$

$b_2 \approx 57.4$. The two solutions are: $A_1 \approx 76.5°$, $B_1 \approx 50.5°, b_1 \approx 111.1$ and $A_2 \approx 103.5°$, $B_2 \approx 23.5°, b_2 \approx 57.4$.

19. Answers will vary.

(i) $h = c \sin B = 20 \sin 60° = 10\sqrt{3}$. For C to have two possible values, $h < b < c$, so choose a value of b such that $10\sqrt{3} < b < 20$.

(ii) C has exactly one value if $c = 10\sqrt{3}$.

(iii) C has no value if $b < c \sin B \Rightarrow b < 10\sqrt{3}$.

20. Answers will vary.

(i) B is an obtuse angle, so it is not possible for there to be two possible values for C.

(ii) C has exactly one value if $b = 60$.

(iii) C has no value if $b < c \sin B \Rightarrow b < 10\sqrt{3}$.

21. Given: $a = 60, b = 90, c = 125$ – an SSS case.

$60^2 = 90^2 + 125^2 - 2(90)(125) \cos A \Rightarrow$

$A \approx 26.6°$

$90^2 = 60^2 + 125^2 - 2(60)(125) \cos B \Rightarrow$

$B \approx 42.1°$

$C = 180° - (26.6° + 42.1°) \approx 111.3°$

22. Given: $a = 15, b = 9, C = 120°$ – an SAS case.

$c^2 = 15^2 + 9^2 - 2(15)(9) \cos 120° \Rightarrow c = 21$

$15^2 = 9^2 + 21^2 - 2(9)(21) \cos A \Rightarrow A \approx 38.2°$

$B \approx 180° - (38.2° + 120°) \approx 21.8°$

23. Given: $a = 40, c = 38, B = 80°$ – an SAS case.

$b^2 = 40^2 + 38^2 - 2(40)(38) \cos 80° \Rightarrow b \approx 50.2$

$\frac{\sin 80°}{50.2} = \frac{\sin A}{40} \Rightarrow$

$A = \sin^{-1}\left(\frac{40 \sin 80°}{50.2}\right) \approx 51.7°$

$C \approx 180° - (80° + 51.7°) \approx 48.3°$

24. Given: $a = 10, b = 20, c = 22$ – an SSS case.
$10^2 = 20^2 + 22^2 - 2(20)(22)\cos A \Rightarrow$
$A \approx 27.012° \approx 27.0°$
$\frac{\sin 27.012°}{10} = \frac{\sin B}{20} \Rightarrow B \approx 65.3°$
$C \approx 180° - (27.0° + 65.3°) \approx 87.7°$

25. Given: $a = 2.6, b = 3.7, c = 4.8$ – an SSS case.
$2.6^2 = 3.7^2 + 4.8^2 - 2(3.7)(4.8)\cos A \Rightarrow$
$A \approx 32.462° \approx 32.5°$
$\frac{\sin 32.462°}{2.6} = \frac{\sin B}{3.7} \Rightarrow B \approx 49.8°$
$C \approx 180° - (32.5° + 49.8°) \approx 97.7°$

26. Given: $a = 15, c = 26, B = 115°$ – an SAS case.
$b^2 = 15^2 + 26^2 - 2(15)(26)\cos 115° \Rightarrow$
$b \approx 35.0805 \approx 35.1$
$\frac{\sin 115°}{35.0805} = \frac{\sin A}{15} \Rightarrow A \approx 22.8°$
$C \approx 180° - (115° + 22.8°) \approx 42.2°$

27. Given: $a = 12, b = 7, C = 130°$ – an SAS case.
$c^2 = 12^2 + 7^2 - 2(12)(7)\cos 130° \Rightarrow$
$c \approx 17.349 \approx 17.3$
$\frac{\sin A}{12} = \frac{\sin 130°}{17.349} \Rightarrow A \approx 32.0°$
$B \approx 180° - (130° + 32.0°) \approx 18.0°$

28. Given: $b = 75, c = 100, A = 80°$ – an SAS case.
$a^2 = 75^2 + 100^2 - 2(75)(100)\cos 80° \Rightarrow$
$a \approx 114.106 \approx 114.1$
$\frac{\sin 80°}{114.106} = \frac{\sin B}{75} \Rightarrow B \approx 40.3°$
$C \approx 180° - (80° + 40.3°) \approx 59.7°$

29. $s = \frac{5 + 7 + 10}{2} = 11$
$k = \sqrt{11(11-5)(11-7)(11-10)} \approx 16\text{ m}^2$

30. $s = \frac{2.4 + 3.4 + 4.4}{2} = 5.1$
$k = \sqrt{5.1(5.1-2.4)(5.1-3.4)(5.1-4.4)} \approx 4\text{ m}^2$

31. $k = \frac{1}{2}bc\sin A = \frac{1}{2}(6)(4)\sin 65° \approx 11\text{ ft}^2$

32. Given: $A = 115°, b = 20, a = 30$
$\frac{\sin 115°}{30} = \frac{\sin B}{20} \Rightarrow B \approx 37.172°$
$C \approx 180° - (115° + 37.172°) \approx 27.828°$
$k = \frac{1}{2}ab\sin C \approx \frac{1}{2}(30)(20)\sin 27.828°$
$\approx 140\text{ in.}^2$

33. $\mathbf{v} = \langle 2-3, 7-5\rangle = \langle -1, 2\rangle = -\mathbf{i} + 2\mathbf{j}$

34. $\mathbf{v} = \langle 5-(-1), (-4-3)\rangle = \langle 6, -7\rangle = 6\mathbf{i} - 7\mathbf{j}$

35. $5\mathbf{v} = 5\langle -2, 3\rangle = \langle -10, 15\rangle$

36. $2\mathbf{v} + \mathbf{w} = 2\langle -2, 3\rangle + \langle 5, -6\rangle = \langle 1, 0\rangle$

37. $3\mathbf{v} - 2\mathbf{w} = 3\langle -2, 3\rangle - 2\langle 5, -6\rangle = \langle -16, 21\rangle$

38. $\|\mathbf{v} + \mathbf{w}\| = \|\langle -2, 3\rangle + \langle 5, -6\rangle\| = \|\langle 3, -3\rangle\|$
$= \sqrt{3^2 + (-3)^2} = 3\sqrt{2}$

39. $\mathbf{v} = \mathbf{i} + \mathbf{j} \Rightarrow \|\mathbf{v}\| = \sqrt{1^2 + 1^2} = \sqrt{2}$
$\mathbf{u} = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$

40. $\mathbf{v} = 2\mathbf{i} - 7\mathbf{j} \Rightarrow \|\mathbf{v}\| = \sqrt{2^2 + (-7)^2} = \sqrt{53}$
$\mathbf{u} = \frac{1}{\sqrt{53}}(2\mathbf{i} - 7\mathbf{j}) = \frac{2\sqrt{53}}{53}\mathbf{i} - \frac{7\sqrt{53}}{53}\mathbf{j}$

41. $\mathbf{v} = \langle 3, -5\rangle \Rightarrow \|\mathbf{v}\| = \sqrt{3^2 + (-5)^2} = \sqrt{34}$
$\mathbf{u} = \frac{1}{\sqrt{34}}\langle 3, -5\rangle = \left\langle \frac{3\sqrt{34}}{34}, -\frac{5\sqrt{34}}{34}\right\rangle$

42. $\mathbf{v} = \langle -5, -2\rangle \Rightarrow \|\mathbf{v}\| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$
$\mathbf{u} = \frac{1}{\sqrt{29}}\langle -5, -2\rangle = \left\langle -\frac{5\sqrt{29}}{29}, -\frac{2\sqrt{29}}{29}\right\rangle$

43. Given: $\|\mathbf{v}\| = 6, \theta = 30°$. Then
$\mathbf{v} = 6(\cos 30°\mathbf{i} + \sin 30°\mathbf{j})$
$= 6\left(\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}\right) = 3\sqrt{3}\mathbf{i} + 3\mathbf{j}$

44. Given: $\|\mathbf{v}\| = 20, \theta = 120°$. Then
$\mathbf{v} = 20(\cos 120°\mathbf{i} + \sin 120°\mathbf{j})$
$= 20\left(-\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}\right) = -10\mathbf{i} + 10\sqrt{3}\mathbf{j}$

45. Given: $\|\mathbf{v}\| = 12, \theta = 225°$. Then

$$\mathbf{v} = 12(\cos 225°\mathbf{i} + \sin 225°\mathbf{j}) = 12\left(-\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}\right) = -6\sqrt{2}\mathbf{i} - 6\sqrt{2}\mathbf{j}$$

46. Given: $\|\mathbf{v}\| = 10, \theta = -30°$. Then

$$\mathbf{v} = 10(\cos(-30°)\mathbf{i} + \sin(-30°)\mathbf{j}) = 10\left(\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}\right) = 5\sqrt{3}\mathbf{i} - 5\mathbf{j}$$

47. $\mathbf{v} \bullet \mathbf{w} = \langle 2, -3\rangle \bullet \langle 3, 4\rangle = 2(3) + (-3)(4) = -6$

$$\text{proj}_{\mathbf{w}}\,\mathbf{v} = \frac{\mathbf{v} \bullet \mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w} = \frac{-6}{3^2 + 4^2}\langle 3, 4\rangle = -\frac{6}{25}\langle 3, 4\rangle = \left\langle -\frac{18}{25}, -\frac{24}{25}\right\rangle$$

48. $\mathbf{v} \bullet \mathbf{w} = \langle -1, -2\rangle \bullet \langle 4, -1\rangle = (-1)(4) + (-2)(-1) = -2$

$$\text{proj}_{\mathbf{w}}\,\mathbf{v} = \frac{\mathbf{v} \bullet \mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w} = \frac{-2}{4^2 + (-1)^2}\langle 4, -1\rangle = -\frac{2}{17}\langle 4, -1\rangle = \left\langle -\frac{8}{17}, \frac{2}{17}\right\rangle$$

49. $\mathbf{v} \bullet \mathbf{w} = \langle 2, -5\rangle \bullet \langle 5, 2\rangle = 2(5) + (-5)(2) = 0$

$$\text{proj}_{\mathbf{w}}\,\mathbf{v} = \frac{\mathbf{v} \bullet \mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w} = \frac{0}{5^2 + 2^2}\langle 5, 0\rangle = 0\langle 5, 0\rangle = \langle 0, 0\rangle$$

50. $\mathbf{v} \bullet \mathbf{w} = \langle 2, 1\rangle \bullet \langle 2, -1\rangle = 2(2) + (1)(-1) = 3$

$$\text{proj}_{\mathbf{w}}\,\mathbf{v} = \frac{\mathbf{v} \bullet \mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w} = \frac{3}{2^2 + (-1)^2}\langle 2, -1\rangle = \frac{3}{5}\langle 2, -1\rangle = \left\langle \frac{6}{5}, -\frac{3}{5}\right\rangle$$

For exercises 51–54, use the formula $\mathbf{v} \bullet \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos\theta$ to find the value of θ.

51. Given: $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}, \mathbf{w} = -\mathbf{i} + 2\mathbf{j}$.

$\|\mathbf{v}\| = \sqrt{2^2 + 3^2} = \sqrt{13}; \|\mathbf{w}\| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$

$\mathbf{v} \bullet \mathbf{w} = (2)(-1) + (3)(2) = 4$

$\mathbf{v} \bullet \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos\theta \Rightarrow 4 = \sqrt{13}\sqrt{5}\cos\theta \Rightarrow$

$\theta = \cos^{-1}\left(\frac{4}{\sqrt{65}}\right) \approx 60.3°$

52. Given: $\mathbf{v} = \mathbf{i} + 4\mathbf{j}, \mathbf{w} = -4\mathbf{i} - \mathbf{j}$.

$\|\mathbf{v}\| = \sqrt{1^2 + 4^2} = \sqrt{17}$

$\|\mathbf{w}\| = \sqrt{(-4)^2 + (-1)^2} = \sqrt{17}$

$\mathbf{v} \bullet \mathbf{w} = (1)(-4) + (4)(-1) = -8$

$\mathbf{v} \bullet \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos\theta \Rightarrow -8 = \sqrt{17}\sqrt{17}\cos\theta \Rightarrow$

$\theta = \cos^{-1}\left(-\frac{8}{17}\right) \approx 118.1°$

53. Given: $\mathbf{v} = \langle 1, 1\rangle, \mathbf{w} = \langle -3, 2\rangle$.

$\|\mathbf{v}\| = \sqrt{1^2 + 1^2} = \sqrt{2}; \|\mathbf{w}\| = \sqrt{(-3)^2 + 2^2} = \sqrt{13}$

$\mathbf{v} \bullet \mathbf{w} = (1)(-3) + (1)(2) = -1$

$\mathbf{v} \bullet \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos\theta \Rightarrow -1 = \sqrt{2}\sqrt{13}\cos\theta \Rightarrow$

$\theta = \cos^{-1}\left(-\frac{1}{\sqrt{26}}\right) \approx 101.3°$

54. Given: $\mathbf{v} = \langle 1, 5\rangle, \mathbf{w} = \langle 3, -1\rangle$.

$\|\mathbf{v}\| = \sqrt{1^2 + 5^2} = \sqrt{26};$

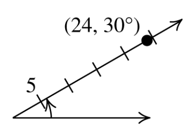
$\|\mathbf{w}\| = \sqrt{3^2 + (-1)^2} = \sqrt{10}$

$\mathbf{v} \bullet \mathbf{w} = (1)(3) + (5)(-1) = -2$

$\mathbf{v} \bullet \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos\theta \Rightarrow -2 = \sqrt{26}\sqrt{10}\cos\theta \Rightarrow$

$\theta = \cos^{-1}\left(-\frac{2}{\sqrt{260}}\right) \approx 97.1°$

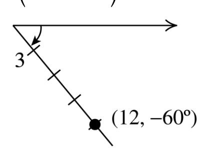
55. $x = 24\cos 30° = 12\sqrt{3}; y = 24\sin 30° = 12$ The rectangular coordinates of $(24, 30°)$ are $(12\sqrt{3}, 12)$



56. $x = 12\cos(-60°) = 12\left(\frac{1}{2}\right) = 6$

$y = 12\sin(-60°) = 12\left(-\frac{\sqrt{3}}{2}\right) = -6\sqrt{3}$

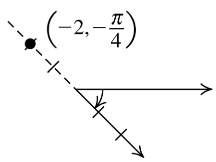
The rectangular coordinates of $(12, -60°)$ are $(6, -6\sqrt{3})$.



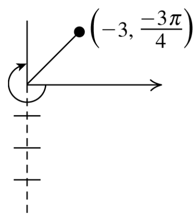
57. $x = -2\cos\left(-\frac{\pi}{4}\right) = -2\left(\frac{\sqrt{2}}{2}\right) = -\sqrt{2}$

$y = -2\sin\left(-\frac{\pi}{4}\right) = -2\left(-\frac{\sqrt{2}}{2}\right) = \sqrt{2}$

The rectangular coordinates of $(-2, -\pi/4)$ are $\left(-\sqrt{2}, \sqrt{2}\right)$.



Exercise 57



Exercise 58

58. $x = -3\cos\left(-\frac{3\pi}{4}\right) = -3\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\sqrt{2}}{2}$

$y = -3\sin\left(-\frac{3\pi}{4}\right) = -3\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\sqrt{2}}{2}$

The rectangular coordinates of $\left(-3, -\frac{3\pi}{4}\right)$ are $\left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$.

59. $r = \sqrt{(-2)^2 + 2^2} = 2\sqrt{2}$

$\tan\theta = \frac{y}{x} = -\frac{2}{2} \Rightarrow \theta = \frac{3\pi}{4}$

The polar form of (−2, 2) is $\left(2\sqrt{2}, \frac{3\pi}{4}\right)$.

60. $r = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = 2$

$\tan\theta = \frac{y}{x} = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$

The polar form of $\left(\sqrt{3}, 1\right)$ is $\left(2, \frac{\pi}{6}\right)$.

61. $r = \sqrt{\left(2\sqrt{3}\right)^2 + (-2)^2} = 4$

$\tan\theta = \frac{y}{x} = -\frac{2}{2\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$ or $\theta = \frac{11\pi}{6}$

$\left(2\sqrt{3}, -2\right)$ lies in Quadrant IV, so $\theta = \frac{11\pi}{6}$.

The polar form of $\left(2\sqrt{3}, -2\right)$ is $\left(4, \frac{11\pi}{6}\right)$.

62. $r = \sqrt{(-2)^2 + \left(-2\sqrt{3}\right)^2} = 4$

$\tan\theta = \frac{y}{x} = \frac{2\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3}$ or $\theta = \frac{4\pi}{3}$

$\left(-2, -2\sqrt{3}\right)$ lies in Quadrant III, so $\theta = \frac{4\pi}{3}$.

The polar form of $\left(-2, -2\sqrt{3}\right)$ is $\left(4, \frac{4\pi}{3}\right)$.

63. $3x + 2y = 12 \Rightarrow 3r\cos\theta + 2r\sin\theta = 12 \Rightarrow$
$r(3\cos\theta + 2\sin\theta) = 12 \Rightarrow r = \frac{12}{3\cos\theta + 2\sin\theta}$

64. $x^2 + y^2 = 36 \Rightarrow r^2 = 36 \Rightarrow r = 6$

65. $x^2 + y^2 = 8x \Rightarrow r^2 = 8r\cos\theta \Rightarrow r = 8\cos\theta$

66. $x^2 + y^2 = 6y \Rightarrow r^2 = 6r\sin\theta \Rightarrow r = 6\sin\theta$

67. $r = -3 \Rightarrow r^2 = 9 \Rightarrow x^2 + y^2 = 9$

68. $\theta = \frac{5\pi}{6} \Rightarrow \tan\theta = \tan\frac{5\pi}{6} \Rightarrow \frac{y}{x} = -\frac{\sqrt{3}}{3} \Rightarrow$
$y = -\frac{\sqrt{3}}{3}x$

69. $r = 3\csc\theta \Rightarrow r = \frac{3}{\sin\theta} \Rightarrow r\sin\theta = 3 \Rightarrow y = 3$

70. $r = 2\sec\theta \Rightarrow r = \frac{2}{\cos\theta} \Rightarrow r\cos\theta = 2 \Rightarrow x = 2$

71. $r = 1 - 2\sin\theta \Rightarrow r + 2\sin\theta = 1 \Rightarrow$
$r^2 + 2r\sin\theta = r \Rightarrow (r^2 + 2r\sin\theta)^2 = r^2 \Rightarrow$
$(x^2 + y^2 + 2y)^2 = x^2 + y^2$

72. $r = 3\cos\theta \Rightarrow r^2 = 3r\cos\theta \Rightarrow x^2 + y^2 = 3x$

73. $-3i = 3\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$

74. $-1 + i \Rightarrow r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$\tan\theta = -1 \Rightarrow \theta = \frac{3\pi}{4}$ or $\theta = \frac{7\pi}{4}$

$-1 + i$ lies in Quadrant II, so $\theta = \frac{3\pi}{4}$.

The polar form of $-1 + i$ is
$\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$.

75. $5\sqrt{3}-5i \Rightarrow r=\sqrt{\left(5\sqrt{3}\right)^2+(-5)^2}=10$

$\tan\theta=-\frac{5}{5\sqrt{3}} \Rightarrow \theta=\frac{5\pi}{6}$ or $\theta=\frac{11\pi}{6}$

$5\sqrt{3}-5i$ lies in Quadrant IV, so $\theta=\frac{11\pi}{6}$.

The polar form of $5\sqrt{3}-5i$ is

$10\left(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right)$.

76. $-2-2\sqrt{3}i \Rightarrow r=\sqrt{(-2)^2+\left(-2\sqrt{3}\right)^2}=4$

$\tan\theta=\frac{-2\sqrt{3}}{-2} \Rightarrow \theta=\frac{\pi}{3}$ or $\theta=\frac{4\pi}{3}$

$-2-2\sqrt{3}i$ lies in Quadrant III, so $\theta=\frac{4\pi}{3}$

The polar form of $-2-2\sqrt{3}i$ is

$4\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)$.

77. $2(\cos 45°+i\sin 45°)=2\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)$
$=\sqrt{2}+\sqrt{2}i$

78. $3(\cos 240°+i\sin 240°)=3\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$
$=-\frac{3}{2}-\frac{3\sqrt{3}}{2}i$

79. $6\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)=6\left(-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)$
$=-3\sqrt{2}+3\sqrt{2}i$

80. $4\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)=4\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)$
$=-2\sqrt{3}-2i$

81. $z_1=3(\cos 25°+i\sin 25°)$
$z_2=2(\cos 10°+i\sin 10°)$
$z_1z_2=6(\cos 35°+i\sin 35°)$
$\frac{z_1}{z_2}=\frac{3}{2}(\cos 15°+i\sin 15°)$

82. $z_1=4(\cos 300°+i\sin 300°)$
$z_2=2(\cos 20°+i\sin 20°)$
$z_1z_2=8(\cos 320°+i\sin 320°)$
$\frac{z_1}{z_2}=2(\cos 280°+i\sin 280°)$

83. $z_1=2\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$
$z_2=3\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$
$z_1z_2=6\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)$
$\frac{z_1}{z_2}=\frac{2}{3}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$

84. $z_1=5\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)$
$z_2=15\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$
$z_1z_2=75\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)$
$\frac{z_1}{z_2}=\frac{1}{3}(\cos\pi+i\sin\pi)$

85. $\left[3(\cos 40°+i\sin 40°)\right]^3$
$=3^3(\cos(3\cdot 40°)+i\sin(3\cdot 40°))$
$=27(\cos 120°+i\sin 120°)$

86. $\left[4\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\right]^6$
$=4^6\left(\cos\left(6\cdot\frac{\pi}{6}\right)+i\sin\left(6\cdot\frac{\pi}{6}\right)\right)$
$=4096(\cos\pi+i\sin\pi)$

87. First, convert $2-2\sqrt{3}i$ to polar form:

$r=\sqrt{2^2+\left(-2\sqrt{3}\right)^2}=4$

$\theta'=\left|\tan^{-1}\left(\frac{-2\sqrt{3}}{2}\right)\right|=\left|-\frac{\pi}{3}\right|=\frac{\pi}{3}$

$2-2\sqrt{3}i$ lies in quadrant IV, so

$\theta=2\pi-\frac{\pi}{3}=\frac{5\pi}{3}$.

The polar form of $2-2\sqrt{3}i$ is

$4\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)$.

$\left[4\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)\right]^6$
$=4^6\left(\cos\left(6\cdot\frac{5\pi}{3}\right)+i\sin\left(6\cdot\frac{5\pi}{3}\right)\right)$
$=4096(\cos 10\pi+i\sin 10\pi)$
$=4096(\cos 0+i\sin 0)$

88. First, convert $2-2i$ to polar form: $r=\sqrt{2^2+(-2)^2}=2\sqrt{2}$

$\theta' = \left|\tan^{-1}\left(\frac{-2}{2}\right)\right| = \left|-\frac{\pi}{4}\right| = \frac{\pi}{4}$

$2-2i$ lies in quadrant IV, so $\theta = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$. Thus, the polar form of $2-2i$ is $2\sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$.

$$\left[2\sqrt{2}\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)\right]^7 = 2^{21/2}\left(\cos\left(7\cdot\frac{7\pi}{4}\right)+i\sin\left(7\cdot\frac{7\pi}{4}\right)\right) = 2^{21/2}\left(\cos\frac{49\pi}{4}+i\sin\frac{49\pi}{4}\right)$$
$$= 2^{21/2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$$

89. The polar form for -125 is $125(\cos 180° + i\sin 180°)$.

$$z_k = 125^{1/3}\cos\frac{180°+360°\cdot k}{3} + i\sin\frac{180°+360°\cdot k}{3} \text{ for } k = 0, 1, 2.$$
$$z_0 = 125^{1/3}\cos\frac{180°}{3}+i\sin\frac{180°}{3} = 5\cos 60° + i\sin 60°$$
$$z_1 = 125^{1/3}\cos\frac{540°}{3}+i\sin\frac{540°}{3} = 5\cos 180° + i\sin 180°$$
$$z_2 = 125^{1/3}\cos\frac{900°}{3}+i\sin\frac{900°}{3} = 5\cos 300° + i\sin 300°$$

90. The polar form for $-16i$ is $16(\cos 270° + i\sin 270°)$

$$z_k = 16^{1/4}\left(\cos\frac{270°+360°\cdot k}{4}+i\sin\frac{270°+360°\cdot k}{4}\right) = 2\left(\cos\frac{270°+360°\cdot k}{4}+i\sin\frac{270°+360°\cdot k}{4}\right)$$

for $k = 0, 1, 2, 3$.

$$z_0 = 2\left(\cos\frac{270°}{4}+i\sin\frac{270°}{4}\right) = 2(\cos 67.5° + i\sin 67.5°)$$
$$z_1 = 2\left(\cos\frac{630°}{4}+i\sin\frac{630°}{4}\right) = 2(\cos 157.5° + i\sin 157.5°)$$
$$z_2 = 2\left(\cos\frac{990°}{4}+i\sin\frac{990°}{4}\right) = 2(\cos 247.5° + i\sin 247.5°)$$
$$z_3 = 2\left(\cos\frac{1350°}{4}+i\sin\frac{1350°}{4}\right) = 2(\cos 337.5° + i\sin 337.5°)$$

91. Write $-1+\sqrt{3}i$ in polar form: $r=\sqrt{(-1)^2+\left(\sqrt{3}\right)^2}=2$; $\theta' = \left|\tan^{-1}\left(-\sqrt{3}\right)\right| = |-60°| = 60°$

Since $-1+\sqrt{3}i$ lies in Quadrant II, $\theta = 180° - 60° = 120°$

$$z_k = 2^{1/5}\left(\cos\frac{120°+360°\cdot k}{5}+i\sin\frac{120°+360°\cdot k}{5}\right) \text{ for } k = 0, 1, 2, 3, 4.$$
$$z_0 = 2^{1/5}\left(\cos\frac{120°}{5}+i\sin\frac{120°}{5}\right) = 2^{1/5}(\cos 24° + i\sin 24°)$$
$$z_1 = 2^{1/5}\left(\cos\frac{480°}{5}+i\sin\frac{480°}{5}\right) = 2^{1/5}(\cos 96° + i\sin 96°)$$
$$z_2 = 2^{1/5}\left(\cos\frac{840°}{5}+i\sin\frac{840°}{5}\right) = 2^{1/5}(\cos 168° + i\sin 168°)$$

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$$z_3 = 2^{1/5}\left(\cos\frac{1200°}{5} + i\sin\frac{1200°}{5}\right) = 2^{1/5}\left(\cos 240° + i\sin 240°\right)$$

$$z_4 = 2^{1/5}\left(\cos\frac{1560°}{5} + i\sin\frac{1560°}{5}\right) = 2^{1/5}\left(\cos 312° + i\sin 312°\right)$$

92. Write $1-i$ in polar form: $r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$; $\theta' = \left|\tan^{-1}(-1)\right| = \left|-45°\right| = 45°$

Since $1-i$ lies in Quadrant IV, $\theta = 360° - 45° = 315°$.

$$z_k = 2^{1/6}\left(\cos\frac{315° + 360° \cdot k}{6} + i\sin\frac{315° + 360° \cdot k}{6}\right) \text{ for } k = 0, 1, 2, 3, 4, 5.$$

$$z_0 = \sqrt{2}^{1/6}\left(\cos\frac{315°}{6} + i\sin\frac{315°}{6}\right) = 2^{1/12}\left(\cos 52.5° + i\sin 52.5°\right)$$

$$z_1 = \sqrt{2}^{1/6}\left(\cos\frac{675°}{6} + i\sin\frac{675°}{6}\right) = 2^{1/12}\left(\cos 112.5° + i\sin 112.5°\right)$$

$$z_2 = \sqrt{2}^{1/6}\left(\cos\frac{1035°}{6} + i\sin\frac{1035°}{6}\right) = 2^{1/12}\left(\cos 172.5° + i\sin 172.5°\right)$$

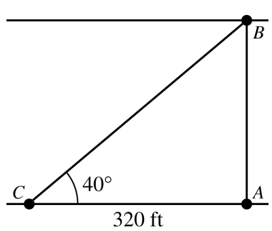
$$z_3 = \sqrt{2}^{1/6}\left(\cos\frac{1395°}{6} + i\sin\frac{1395°}{6}\right) = 2^{1/12}\left(\cos 232.5° + i\sin 232.5°\right)$$

$$z_4 = \sqrt{2}^{1/6}\left(\cos\frac{1755°}{6} + i\sin\frac{1755°}{6}\right) = 2^{1/12}\left(\cos 292.5° + i\sin 292.5°\right)$$

$$z_5 = \sqrt{2}^{1/6}\left(\cos\frac{2115°}{6} + i\sin\frac{2115°}{6}\right) = 2^{1/12}\left(\cos 352.5° + i\sin 352.5°\right)$$

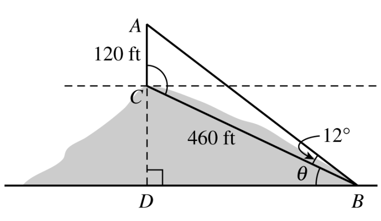
Applying the Concepts

93.



$$\tan 40° = \frac{AB}{320} \Rightarrow AB \approx 268.5 \text{ ft}$$

94.



In $\triangle ABC, \dfrac{\sin 12°}{120} = \dfrac{\sin A}{460} \Rightarrow A \approx 52.8°$

$m\angle ACB \approx 180° - (12° + 52.8°) \approx 115.2°$

In $\triangle CDB, m\angle DCB \approx 180° - 115.2° \approx 64.8°$

$m\angle DCB + \theta = 90° \Rightarrow \theta \approx 90° - 64.8° \approx 25.2°$

95. After 80 minutes, the cars have traveled

$55\left(\dfrac{80}{60}\right) \approx 73.33$ mi and $65\left(\dfrac{80}{60}\right) \approx 86.67$ mi,

respectively. The distance between the two cars is the length of the third side of the triangle that is formed by the roads. Using the Law of Cosines, we have

$$c = \sqrt{73.33^2 + 86.67^2 - 2(73.33)(86.67)\cos 72°} \approx 94.7 \text{ miles}$$

96. In a triangle, the largest angle is opposite the longest side. So the largest angle is opposite the side that is 415 feet. Call that side c, then use the Law of Cosines to find C:

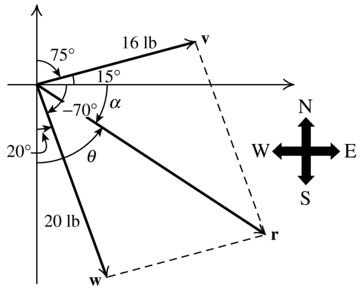
$$415^2 = 175^2 + 310^2 - 2(175)(310)\cos C \Rightarrow C \approx 114.8°$$

97. Using Heron's Formula, we have

$$s = \frac{310 + 415 + 175}{2} = 450$$

$$k = \sqrt{450(450 - 310)(450 - 415)(450 - 175)} \approx 24{,}625 \text{ ft}^2$$

98.



$\mathbf{v} = 16(\cos 15°\mathbf{i} + \sin 15°\mathbf{j}) \approx 15.45\mathbf{i} + 4.14\mathbf{j}$
$\mathbf{w} = 20(\cos(-70°)\mathbf{i} + \sin(-70°)\mathbf{j})$
$= 20(\cos 70°\mathbf{i} - \sin 70°\mathbf{j}) \approx 6.84\mathbf{i} - 18.79\mathbf{j}$
$\mathbf{r} = \mathbf{v} + \mathbf{w} \approx 22.29\mathbf{i} - 14.65\mathbf{j} \approx 22.3\mathbf{i} - 14.7\mathbf{j}$
$\|\mathbf{r}\| = \sqrt{22.29^2 + (-14.65)^2} \approx 26.7$ pounds
$\alpha = \tan^{-1}\left(-\dfrac{14.65}{22.29}\right) \approx -33.314° \Rightarrow$
$\theta \approx 90° - 33.314° \approx 56.7°$
The magnitude of the resultant is 26.7 pounds and its bearing is S 56.7° E.

99. $W = F \bullet \overrightarrow{PQ} = \|F\|\|\overrightarrow{PQ}\|\cos\theta$
$= 30(60)\cos 48° \approx 1204.4$ foot-pounds

100. $Z_1 = 20\left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right)$
$= 20(\cos 30° + i\sin 30°) = 10\sqrt{3} + 10i$
$Z_2 = 10\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)$
$= 10(\cos 120° + i\sin 120°) = -5 + 5\sqrt{3}i$
$Z_1Z_2 = 200(\cos 150° + i\sin 150°)$
$Z_1 + Z_2 = (10\sqrt{3} - 5) + (10 + 5\sqrt{3})i$.
Write $Z_1 + Z_2$ in polar form:
$r = \sqrt{(10\sqrt{3}-5)^2 + (10+5\sqrt{3})^2} \approx 22.36 \approx 10\sqrt{5}$
$\theta' = \left|\tan^{-1}\left(\dfrac{10+5\sqrt{3}}{10\sqrt{3}-5}\right)\right| \approx 56.5°$. So the polar form for $Z_1 + Z_2$ is $10\sqrt{5}(\cos 56.5° + i\sin 56.5°)$
$\dfrac{Z_1Z_2}{Z_1+Z_2} \approx \dfrac{200(\cos 150° + i\sin 150°)}{10\sqrt{5}(\cos 56.5° + i\sin 56.5°)}$
$\approx 4\sqrt{5}(\cos 93.5° + i\sin 93.5°)$

Chapter 7 Practice Test A

1. Given: $A = 115°, C = 35°, c = 15$ − an AAS case. $B = 180° - (115° + 35°) = 30°$
$\dfrac{a}{\sin 115°} = \dfrac{15}{\sin 35°} \Rightarrow a \approx 23.7$
$\dfrac{b}{\sin 30°} = \dfrac{15}{\sin 35°} \Rightarrow b \approx 13.1$

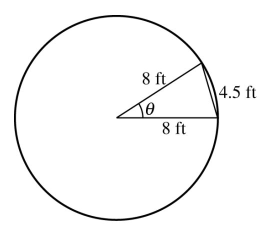
2. Given: $A = 42°, B = 37°, a = 50$ m − an AAS case. $C = 180° - (42° + 37°) = 101°$
$\dfrac{50}{\sin 42°} = \dfrac{b}{\sin 37°} \Rightarrow b \approx 45.0$ m
$\dfrac{50}{\sin 42°} = \dfrac{c}{\sin 101°} \Rightarrow c \approx 73.4$ m

3. Given: $a = 35, B = 106°, c = 53$ − an SAS case.
$b^2 = 35^2 + 53^2 - 2(35)(53)\cos 106° \Rightarrow$
$b \approx 71.11 \approx 71.1$
$\dfrac{\sin 106°}{71.11} = \dfrac{\sin A}{35} \Rightarrow A \approx 28.2°$
$C \approx 180° - (106° + 28.2°) \approx 45.8°$

4. Given: $a = 30$ ft, $b = 20$ ft, $c = 25$ ft − an SSS case. $30^2 = 20^2 + 25^2 - 2(20)(25)\cos A \Rightarrow$
$A \approx 82.819° \approx 82.8$
$\dfrac{\sin 82.819°}{30} = \dfrac{\sin B}{20} \Rightarrow B \approx 41.4°$
$C \approx 180° - (82.8° + 41.4°) \approx 55.8°$

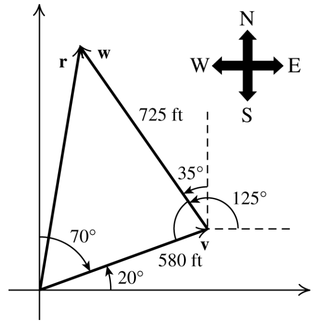
5. Given: $a = 5.6, b = 4.1$.
Since the triangle is a right triangle, use the Pythagorean theorem to find c:
$c = \sqrt{a^2 + b^2} = \sqrt{5.6^2 + 4.1^2} \approx 6.9$
$A = \tan^{-1}\left(\dfrac{a}{b}\right) = \tan^{-1}\left(\dfrac{5.6}{4.1}\right) \approx 53.8°$
$B = 90 - A = 90 - 53.8° = 36.2°$

6.



$4.5^2 = 8^2 + 8^2 - 2(8)(8)\cos\theta \Rightarrow \theta \approx 32.7°$

7.



$\mathbf{v} = 580(\cos 20°\mathbf{i} + \sin 20°\mathbf{j}) \approx 545.0\mathbf{i} + 198.4\mathbf{j}$
$\mathbf{w} = 725(\cos 125°\mathbf{i} + \sin 125°\mathbf{j})$
$\approx -415.8\mathbf{i} + 593.9\mathbf{j}$
$\mathbf{r} = \mathbf{v} + \mathbf{w} = 129.2\mathbf{i} + 792.3\mathbf{j}$
$\|\mathbf{r}\| = \sqrt{129.2^2 + 792.3^2} \approx 803$ feet

8. $\mathbf{v} = \langle 2-3, -7-5 \rangle = \langle -1, -12 \rangle$

9. $\mathbf{v} - \mathbf{w} = \langle -2-1, 3-5 \rangle = \langle -3, -2 \rangle$

10. $2\mathbf{v} - 3\mathbf{w} = 2(-2\mathbf{i} + 3\mathbf{j}) - 3(\mathbf{i} + 5\mathbf{j}) = -7\mathbf{i} - 9\mathbf{j}$

11. $v_1 = \|\mathbf{v}\|\cos\theta = 3\cos(-30°) = \frac{3\sqrt{3}}{2}$
$v_2 = \|\mathbf{v}\|\sin\theta = 3\sin(-30°) = -\frac{3}{2}$
$\mathbf{v} = \frac{3\sqrt{3}}{2}\mathbf{i} - \frac{3}{2}\mathbf{j}$

12. $\mathbf{v} \bullet \mathbf{w} = (4\mathbf{i} + 3\mathbf{j}) \bullet (-\mathbf{i} + 7\mathbf{j}) = -4 + 3(7) = 17$

13. Given: $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}, \mathbf{w} = -2\mathbf{i} + 5\mathbf{j}$

$$\cos\theta = \frac{\mathbf{v} \bullet \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{(3)(-2) + (-4)(5)}{\left(\sqrt{3^2 + (-4)^2}\right)\left(\sqrt{(-2)^2 + 5^2}\right)}$$
$$= -\frac{26}{5\sqrt{29}} \Rightarrow \theta = \cos^{-1}\left(-\frac{26}{5\sqrt{29}}\right) \approx 164.9°$$

14. $x = -2\cos(-45°) = -\sqrt{2}; y = -2\sin(-45°) = \sqrt{2}$
The rectangular coordinates for $(-2, -45°)$ are $\left(-\sqrt{2}, \sqrt{2}\right)$.

15. $r = \sqrt{\left(-\sqrt{3}\right)^2 + (-1)^2} = 2; \tan\theta = \frac{1}{\sqrt{3}} \Rightarrow$
$\theta = \frac{\pi}{6}$ or $\theta = \frac{7\pi}{6}$. $\left(-\sqrt{3}, -1\right)$ is in Quadrant III, so $\theta = \frac{7\pi}{6}$. The polar coordinates for $\left(-\sqrt{3}, -1\right)$ are $\left(2, \frac{7\pi}{6}\right)$.

16. $r = -3 \Rightarrow r^2 = 9 \Rightarrow x^2 + y^2 = 9$. The curve is a circle with center (0, 0) and radius 3.

17. $$3\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right) = 3\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$
$$= -\frac{3}{2} + \frac{3\sqrt{3}}{2}i$$

18. First write both expressions in polar form:
$z_1 = 1 - \sqrt{3}i \Rightarrow r = \sqrt{1^2 + \left(-\sqrt{3}\right)^2} = 2$
$\tan\theta = -\sqrt{3} \Rightarrow \theta = 120°$ or $\theta = 300°$
$1 - \sqrt{3}i$ lies in Quadrant IV, so $\theta = 300°$
$z_1 = 2(\cos 300° + i\sin 300°)$
$z_2 = -1 + i \Rightarrow r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$
$\theta' = \left|\tan^{-1}(-1)\right| \Rightarrow \theta' = 45°$
$-1 + i$ lies in Quadrant II, so
$\theta = 180° - 45° = 135°$.
$z_2 = \sqrt{2}(\cos 135° + i\sin 135°)$
$$\frac{z_1}{z_2} = \frac{2(\cos 300° + i\sin 300°)}{\sqrt{2}(\cos 135° + i\sin 135°)}$$
$$= \sqrt{2}(\cos 165° + i\sin 165°)$$

19. $$\left(\sqrt{5}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)^{-6}$$
$$= 5^{-3}\left(\cos\left(-\frac{3\pi}{2}\right) + i\sin\left(-\frac{3\pi}{2}\right)\right) = \frac{1}{125}i$$

20. First write $1 + i$ in polar form:
$r = \sqrt{1^2 + 1^2} = \sqrt{2}; \quad \theta' = \tan^{-1}(1) \Rightarrow \theta' = 45°$
$1 + i$ lies in Quadrant I, so $\theta = 45°$.
The polar coordinates for $1 + i$ are $\sqrt{2}(\cos 45° + i\sin 45°)$.

$$z_k = \left(\sqrt{2}\right)^{1/4}\left(\cos\frac{45° + 360° \cdot k}{4} + i\sin\frac{45° + 360° \cdot k}{4}\right)$$
$$= 2^{1/8}\left(\cos\frac{45° + 360° \cdot k}{4} + i\sin\frac{45° + 360° \cdot k}{4}\right)$$
for $k = 0, 1, 2, 3$.
$$z_0 = 2^{1/8}\left(\cos\frac{45°}{4} + i\sin\frac{45°}{4}\right)$$
$$= 2^{1/8}(\cos 11.25° + i\sin 11.25°)$$

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$$z_1 = 2^{1/8}\left(\cos\frac{405°}{4} + i\sin\frac{405°}{4}\right)$$
$$= 2^{1/8}(\cos 101.25° + i\sin 101.25°)$$
$$z_2 = 2^{1/8}\left(\cos\frac{765°}{4} + i\sin\frac{765°}{4}\right)$$
$$= 2^{1/8}(\cos 191.25° + i\sin 191.25°)$$
$$z_3 = 2^{1/8}\left(\cos\frac{1125°}{4} + i\sin\frac{1125°}{4}\right)$$
$$= 2^{1/8}(\cos 281.25° + i\sin 281.25°)$$

Chapter 7 Practice Test B

1. Given: $A = 105°, C = 35°, c = 12$ − an AAS case. $B = 180° - (105° + 35°) = 40°$

$$\frac{b}{\sin 40°} = \frac{12}{\sin 35°} \Rightarrow b \approx 13.4$$
The answer is A.

2. Given: $A = 27°, B = 50°, a = 25$ m − an AAS case. $C = 180° - (27° + 50°) = 103°$

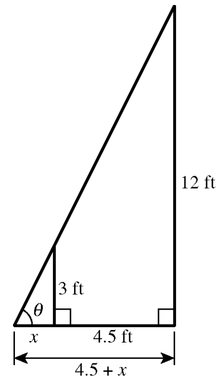
$$\frac{c}{\sin 103°} = \frac{25}{\sin 27°} \Rightarrow c \approx 53.7 \text{ m}$$
The answer is C.

3. Given: $A = 25°, a = 25$ ft, $b = 30$ ft − an SSA case. A is acute, so examine h:
$h = b\sin A = 30\sin 25° \approx 12.7 \Rightarrow h < a < b,$ so there are two triangles.

$$\frac{\sin 25°}{25} = \frac{\sin B}{30} \Rightarrow B_1 \approx 30.473°, B_2 \approx 149.527°$$
$$C_1 \approx 180° - (25° + 30.5°) \approx 124.527°$$
$$\frac{25}{\sin 25°} = \frac{c_1}{\sin 124.527°} \Rightarrow c_1 \approx 48.7$$
$$C_2 \approx 180° - (25° + 149.527°) \approx 5.473°$$
$$\frac{25}{\sin 25°} = \frac{c_2}{\sin 5.473°} \Rightarrow c_2 \approx 5.6$$
The answer is C.

4. Given: $B = 110°, a = 37, c = 21$ − an SAS case.
$b^2 = 37^2 + 21^2 - 2(37)(21)\cos 110° \Rightarrow$ The
$b \approx 48.4$
answer is D.

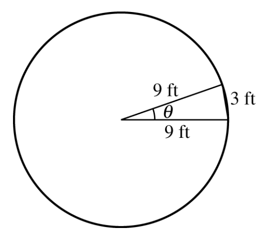
5.



$$\tan\theta = \frac{12}{45 + x} = \frac{3}{x} \Rightarrow x = 1.5 \text{ ft}$$
The answer is A.

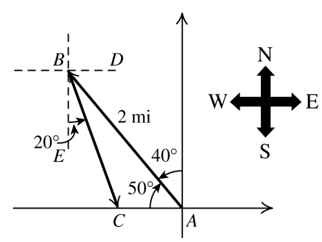
6. Given: $a = 12$ ft, $b = 13$ ft, $c = 7$ ft − an SSS case. $12^2 = 13^2 + 7^2 - 2(13)(7)\cos A \Rightarrow$
$A \approx 66°$. The answer is B.

7.



$3^2 = 9^2 + 9^2 - 2(9)(9)\cos\theta \Rightarrow \theta \approx 19.2°$.
The answer is D.

8.



$m\angle ABD = m\angle BAC = 50°$
$m\angle BAC + m\angle ABC + m\angle CBE = 90° \Rightarrow$
$m\angle ABC = 90° - 20° - 50° = 20°$
$m\angle BCA = 180° - (20° + 50°) = 110°$

$$\frac{2}{\sin 110°} = \frac{BC}{\sin 50°} \Rightarrow BC \approx 1.6 \text{ mi}.$$
The answer is C.

9. $\mathbf{v} = \langle 3 - (-2), -1 - 5\rangle = \langle 5, -6\rangle$
The answer is A.

10. $\mathbf{v} - \mathbf{w} = \langle 1 - 3, -4 - (-2)\rangle = \langle -2, -2\rangle$
The answer is C.

11. $2\mathbf{v} - 3\mathbf{w} = 2(7\mathbf{i} + 3\mathbf{j}) - 3(-2\mathbf{i} + \mathbf{j}) = 20\mathbf{i} + 3\mathbf{j}$
The answer is A.

12. $v_1 = 6\cos(-60°) = 3; v_2 = 6\sin(-60°) = -3\sqrt{3}$
$\mathbf{v} = 3\mathbf{i} - 3\sqrt{3}\mathbf{j}$. The answer is D.

13. $\mathbf{v} \bullet \mathbf{w} = (-2)(9) + (-5)(-4) = 2$.
The answer is D.

14. $\cos\theta = \frac{\mathbf{v} \bullet \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{7-\sqrt{2}}{\left(\sqrt{7^2+(-1)^2}\right)\left(\sqrt{1^2+\left(\sqrt{2}\right)^2}\right)}$
$= \frac{7-\sqrt{2}}{\left(5\sqrt{2}\right)\sqrt{3}} = \frac{7-\sqrt{2}}{5\sqrt{6}} \Rightarrow$
$\theta = \cos^{-1}\left(\frac{7-\sqrt{2}}{5\sqrt{6}}\right) \approx 62.9° \approx 63°$
The answer is B.

15. $x = -4\cos(-30°) = -2\sqrt{3}$
$y = -4\sin(-30°) = 2$
The rectangular coordinates for $(-4, -30°)$ are $\left(-2\sqrt{3}, 2\right)$. The answer is C.

16. $r = \sqrt{3^2 + \left(-\sqrt{3}\right)^2} = 2\sqrt{3}$
$\tan\theta = -\frac{\sqrt{3}}{3} \Rightarrow \theta = 150°$ or $\theta = 330°$
$\left(3, -\sqrt{3}\right)$ is in Quadrant IV, so $\theta = 330°$.
The polar coordinates for $\left(3, -\sqrt{3}\right)$ are $\left(2\sqrt{3}, 330°\right)$. The answer is B.

17. The answer is B.
$r = -2 \Rightarrow r^2 = 4 \Rightarrow x^2 + y^2 = 4$

18. $5\left(\cos\frac{7\pi}{3} + i\sin\frac{7\pi}{3}\right) = 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$
$= 5\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$
$= \frac{5}{2} + \frac{5\sqrt{3}}{2}i$
The answer is C.

19. $z_1 = i \Rightarrow z_1 = \cos 90° + i\sin 90°$
$z_2 = 1 - i \Rightarrow r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$
$\theta' = \left|\tan^{-1}(-1)\right| = 45°$
$1 - i$ is in Quadrant IV, so
$\theta = 360° - 45° = 315°$.
$z_2 = \sqrt{2}(\cos 315° + i\sin 315°)$

$\frac{z_1}{z_2} = \frac{\cos 90° + i\sin 90°}{\sqrt{2}(\cos 315° + i\sin 315°)}$
$= \frac{\sqrt{2}}{2}\left(\cos(-225°) + i\sin(-225°)\right)$
$= \frac{\sqrt{2}}{2}(\cos 135° + i\sin 135°)$
The answer is C.

20. $\left[\sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)\right]^4$
$= \left(\sqrt{2}\right)^4\left(\cos\left(4\cdot\frac{\pi}{12}\right) + i\sin\left(4\cdot\frac{\pi}{12}\right)\right)$
$= 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 4\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$
$= 2 + 2\sqrt{3}i$
The answer is C.

Cumulative Review Exercises Chapters P–7

1. $f(x) = \sqrt{x};\ g(x) = \frac{1}{x-2}$
$\frac{g}{f} = \frac{\frac{1}{x-2}}{\sqrt{x}} = \frac{1}{(x-2)\sqrt{x}}$
f is not defined for $x < 0$, while g is not defined for $x = 2$. $\frac{g}{f}$ is not defined for $x \le 0$ or $x = 2$, so the domain of $\frac{g}{f}$ is $(0, 2) \cup (2, \infty)$.

2. $f(x) = 2x + 7 \Rightarrow y = 2x + 7$
Interchange x and y, then solve for y:
$x = 2y + 7 \Rightarrow y = \frac{x-7}{2} \Rightarrow f^{-1}(x) = \frac{x}{2} - \frac{7}{2}$.

3. The graph of $f(x) = -2x^2 + 8x + 10$ is a parabola with $a = -2$, $b = 8$ and $c = 10$. The parabola opens downward because $a < 0$.
Now, find the vertex:
$h = -\frac{b}{2a} = -\frac{8}{2(-2)} = 2$
$k = f(h) = f(2) = -2(2)^2 + 8(2) + 10 = 18$
Thus, the vertex (h, k) is (2, 18).

(*continued on next page*)

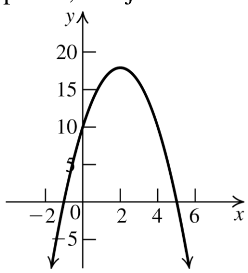
(*continued*)

Next, find the *x*-intercepts:

$-2x^2+8x+10=0 \Rightarrow -2\left(x^2-4x-5\right)=0 \Rightarrow -2(x-5)(x+1)=0 \Rightarrow x=5$ or $x=-1$

Now, find the *y*-intercept: $f(0)=-2(0)^2+8(0)+10=10$.

Thus, the intercepts are (–1, 0), (5, 0) and (0, 10). Use the fact that the parabola is symmetric with respect to its axis, $x=2$, to locate additional points. Plot the vertex, the *x*-intercepts, the *y*-intercept, and any additional points, and join them with a parabola.



4. First solve the associated equation $x^2-6x+8=0.$

$x^2-6x+8=0 \Rightarrow (x-4)(x-2)=0 \Rightarrow x=4$ or $x=2.$

Choose a value of *x* between 2 and 4 (i.e., 3) to test.

$3^2-6(3)+8\overset{?}{>}0$

$-1<0$

Thus, 3 is not in the solution set. The solution set is $(-\infty,2)\cup(4,\infty)$.

5. $\sin\theta\cos\theta=-\frac{\sqrt{3}}{4} \Rightarrow 2\sin\theta\cos\theta=-\frac{\sqrt{3}}{2} \Rightarrow \sin 2\theta=-\frac{\sqrt{3}}{2} \Rightarrow 2\theta\in\left\{\frac{4\pi}{3},\frac{5\pi}{3},\frac{10\pi}{3},\frac{11\pi}{3}\right\} \Rightarrow$

$\theta\in\left\{\frac{2\pi}{3},\frac{5\pi}{6},\frac{5\pi}{3},\frac{11\pi}{6}\right\}$

6. $4\sin^2\theta-1=0 \Rightarrow (2\sin\theta-1)(2\sin\theta+1)=0 \Rightarrow \sin\theta=\frac{1}{2}$ or $\sin\theta=-\frac{1}{2}$. If $\sin\theta=\frac{1}{2}$, $\theta\in\left\{\frac{\pi}{6},\frac{5\pi}{6}\right\}$.

If $\sin\theta=-\frac{1}{2}, \theta\in\left\{\frac{7\pi}{6},\frac{11\pi}{6}\right\}$. The solution set is $\left\{\frac{\pi}{6},\frac{5\pi}{6},\frac{7\pi}{6},\frac{11\pi}{6}\right\}$.

7. Write $\sqrt{3}+i$ in polar form:

$r=\sqrt{\left(\sqrt{3}\right)^2+1}=2;\ \theta'=\left|\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right| \Rightarrow \theta'=30°.$ $\sqrt{3}+i$ lies in Quadrant I, so $\theta=30°$.

The polar coordinates for $\sqrt{3}+i$ are $2(\cos 30°+i\sin 30°)$.

$z_k=2^{1/4}\left(\cos\frac{30°+360°\cdot k}{4}+i\sin\frac{30°+360°\cdot k}{4}\right)$ for k = 0, 1, 2, 3.

$z_0=2^{1/4}\left(\cos\frac{30°}{4}+i\sin\frac{30°}{4}\right)=2^{1/4}\left(\cos 7.5°+i\sin 7.5°\right)$

$z_1=2^{1/4}\left(\cos\frac{390°}{4}+i\sin\frac{390°}{4}\right)=2^{1/4}\left(\cos 97.5°+i\sin 97.5°\right)$

$z_2=2^{1/4}\left(\cos\frac{750°}{4}+i\sin\frac{750°}{4}\right)=2^{1/4}\left(\cos 187.5°+i\sin 187.5°\right)$

$z_3=2^{1/4}\left(\cos\frac{1110°}{4}+i\sin\frac{1110°}{4}\right)=2^{1/4}\left(\cos 277.5°+i\sin 277.5°\right)$

8. Write −64 in polar form: $-64 = 64(\cos 180° + i\sin 180°)$

$$z_k = 64^{1/6}\left(\cos\frac{180° + 360°\cdot k}{6} + i\sin\frac{180° + 360°\cdot k}{6}\right) = 2\left(\cos\frac{180° + 360°\cdot k}{6} + i\sin\frac{180° + 360°\cdot k}{6}\right)$$ for $k = 0,$ 1, 2, 3, 4, 5.

$$z_0 = 2\left(\cos\frac{180°}{6} + i\sin\frac{180°}{6}\right) = 2\left(\cos 30° + i\sin 30°\right)$$

$$z_1 = 2\left(\cos\frac{540°}{6} + i\sin\frac{540°}{6}\right) = 2\left(\cos 90° + i\sin 90°\right)$$

$$z_2 = 2\left(\cos\frac{900°}{6} + i\sin\frac{900°}{6}\right) = 2\left(\cos 150° + i\sin 150°\right)$$

$$z_3 = 2\left(\cos\frac{1260°}{6} + i\sin\frac{1260°}{6}\right) = 2\left(\cos 210° + i\sin 210°\right)$$

$$z_4 = 2\left(\cos\frac{1620°}{6} + i\sin\frac{1620°}{6}\right) = 2\left(\cos 270° + i\sin 270°\right)$$

$$z_5 = 2\left(\cos\frac{1980°}{6} + i\sin\frac{1980°}{6}\right) = 2\left(\cos 330° + i\sin 330°\right)$$

9. First find the slope of the line $2x + 3y + 6 = 0$:

$$2x + 3y + 6 = 0 \Rightarrow y = -\frac{2}{3}x - 2 \Rightarrow m = -\frac{2}{3}.$$

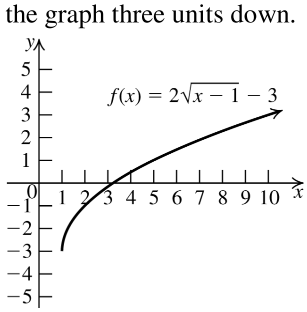
So the slope of the perpendicular line is $\frac{3}{2}$.

The equation of the perpendicular line is

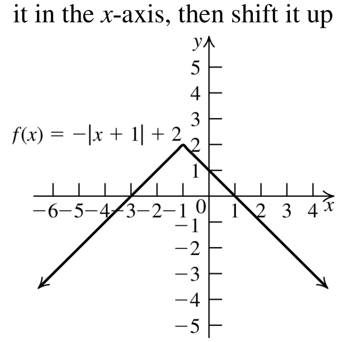
$$y - 5 = \frac{3}{2}(x+1) \Rightarrow y = \frac{3}{2}x + \frac{13}{2}.$$

10. $$2\log x + \frac{1}{2}\log(y+1) - \log(3x+1) = \log\frac{x^2\sqrt{y+1}}{3x+1}$$

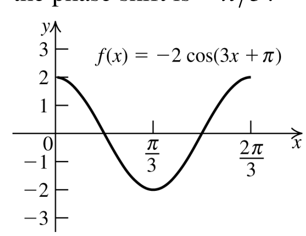
11. Shift the graph of $y = \sqrt{x}$ one unit right, stretch it vertically by a factor of 2, then shift the graph three units down.



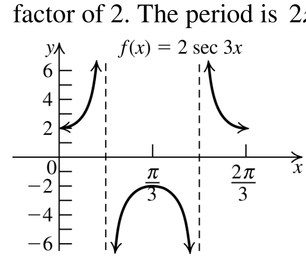
12. Shift the graph of $y = |x|$ one unit left, reflect it in the *x*-axis, then shift it up two units.



13. Reflect the graph of $y = \cos x$ in the *x*-axis. The amplitude is 2, the period is $2\pi/3$, and the phase shift is $-\pi/3$.



14. Stretch the graph of $y = \sec x$ vertically by a factor of 2. The period is $2\pi/3$.



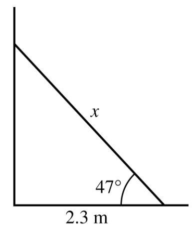
15. $\cos\theta = -\frac{5}{13} = \frac{x}{r} \Rightarrow r = 13, x = -5 \Rightarrow y = 12 \Rightarrow$
$\tan\theta = -\frac{12}{5}$

16. $\cos 65° \cos 35° + \sin 65° \sin 35°$
$= \cos(65° - 35°) = \cos 30° = \frac{\sqrt{3}}{2}$

17. $\frac{1}{1-\cos x} + \frac{1}{1+\cos x} = \frac{1+\cos x + 1 - \cos x}{1-\cos^2 x}$
$= \frac{2}{\sin^2 x} = 2\csc^2 x$
$= 2(1+\cot^2 x)$

18. $\frac{1+\cos 2x}{\sin 2x} = \frac{1+(2\cos^2 x - 1)}{2\sin x \cos x} = \frac{2\cos^2 x}{2\sin x \cos x}$
$= \frac{\cos x}{\sin x} = \cot x$

19. Given: $A = 65°, B = 46°, c = 60$ m – an ASA case. $C = 180° - (65° + 46°) = 69°$
$\frac{60}{\sin 69°} = \frac{a}{\sin 65°} \Rightarrow a \approx 58.2$ m
$\frac{60}{\sin 69°} = \frac{b}{\sin 46°} \Rightarrow b \approx 46.2$ m

20.


$\cos 47° = \frac{2.3}{x} \Rightarrow x \approx 3.4$ m